Adapting Logic to Physics

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Outline

- Physics and Logic
- Boole’s development and logical diagrams
- Operators in logic
- Eigenlogic
- Many-valued Eigenlogic
- Logical syntax and semantics in quantum operators
- Fuzzy Eigenlogic
- Logical interpretation of the Bell Inequalities
- Quantum computing end Eigenlogic
- Quantum Robots

Physics & Logic

Representing Physics

Physical quantities are represented in various mathematical forms:

- **Scalars** (mass, energy, wavelength…)
- **Vectors** (momentum, electromagnetic field,…)
- **Matrices** (spin, Hamiltonians, optical transformations…)
- **Tensors** (gravitation, stress…)

So restricting the **language** in physics to **Boolean functions** of the **two numbers 0 and** 1 is very hindering.

A basic aspect of quantum physics is ***quantification*** represented by the ***spectrum***

corresponding to the eigenvalues of **operators** (matrices) called **observables**.

<u>A quantum measurement gives one of the eigenvalues with a certain probability</u>.

- **Angular Momentum**: discrete (positive and negative) and finite; e.g. $\{-\frac{1}{2}, +\frac{1}{2}\}$ (Fermions) or $\{-1, 0, +1\}$

- **Harmonic Oscillator**: discrete positive and infinite. $\{0,1,2,\cdots\}$ (Bosons)

- **Position and Momentum**: continuous

…

Why adapting logic ?

Greek word **logos**, means both **speech** and **reason**,

Logic defines languages whose **syntax** constructs formal languages and whose **semantics** interprets them.

Syntax should have a link with the structure of Physics (symmetry, geometry, particles, waves…)

Semantics should correspond to the valuation (measurement) in Physics (discrete, rational, complex, random…)

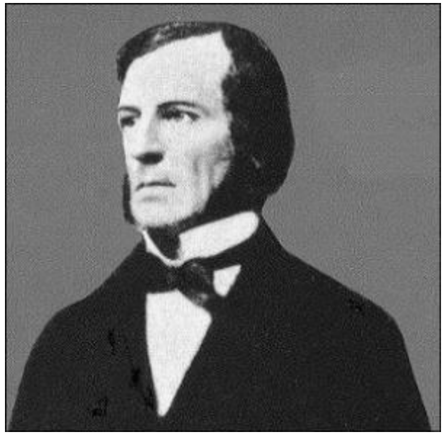
Looking for more "physical" representations of logic. How ?

Different possibilities:

- Representing Boolean logic {0,1} by **operators** (quantum logic, quantum computing, …)
- Using alternative binary logical values, **negative numbers** e.g. $\{-1,+1\}$, (angular momentum, Ising model…)
- Using **many-valued** logic (Qudits, Fourier Transform,…)
- Using **fuzzy** logic (quantum probabilistic interpretation…)
- Representing the logic by operators instead of functions to handle with **noncommutativity** and **reversibility**

George Boole

George Boole : the truth values “ 0 ” and “ 1 ”



George Boole in 1847 [a] gave a mathematical symbolism for logical propositions.

The **conjunction** (AND) of 2 logical propositions X and Y is the product:

$$xy = yx$$

Thus x (“elective” symbol) acts as a selection operator on y (also y on x)

applied on itself the proposition does not change: $x^2 = x$

also written as $x(1 - x) = 0$: the principle of non contradiction

showing that x is **idempotent** and orthogonal to $(1 - x)$

the solutions of this equation are the numbres 0 and 1 representing “False" and “True“ respectively.

This equation was considered by George Boole the “**fundamental law of thought**”! [b]

The method was extended by Boole in the continuous interval [0,1] to give one of the first mathematical formalizations of probabilities in [b].

[a] Boole, G.: **The Mathematical Analysis of Logic. Being an Essay To a Calculus of Deductive Reasoning**, (1847)
[b] Boole, G.: **An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities**, Macmillan, (1854)

Boole's method for generating logical functions and truth tables [*]

Boole's idempotent logical functions $f \in \{0,1\}$ are expressed in an arithmetical form (not modulo 2!)

a logical function of two variables can be expressed by a **bilinear form** of symbols x and y and **truth values** $f(a,b)$

$$f = f(0,0)(1-x)(1-y) + f(0,1)(1-x)y + f(1,0)x(1-y) + f(1,1)xy$$

Negation is the **complementation** by subtracting f from the number 1: $\bar{f} = 1 - f$

generalizes to any number of arguments

TABLE 1. The four single argument logical elective functions

Function $f_i^{[1]}$	Operator	Truth table	Canonical form	Arithmetic form
$f_0^{[1]}$	F	0 0	0	0
$f_1^{[1]}$	$\bar{A}$	1 0	$(1-x)$	$1-x$
$f_2^{[1]}$	A	0 1	x	x
$f_3^{[1]}$	T	1 1	$(1-x)+x$	1

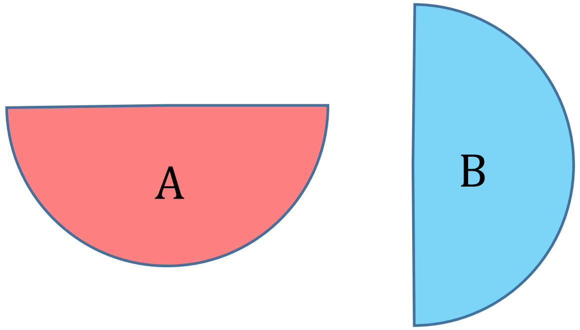
TABLE 2. The sixteen two argument logical elective functions

Funct. $f_i^{[2]}$	Connective for A and B	Truth table	Canonical form	Arithmetic form
$f_0^{[2]}$	F	0 0 0 0	0	0
$f_1^{[2]}$	NOR , $\bar{A} \wedge \bar{B}$	1 0 0 0	$(1-x)(1-y)$	$1-x-y+xy$
$f_2^{[2]}$	$\overline{A \Leftarrow B}$	0 1 0 0	$(1-x)y$	$y-xy$
$f_3^{[2]}$	$\bar{A}$	1 1 0 0	$(1-x)(1-y)+(1-x)y$	$1-x$
$f_4^{[2]}$	$\overline{A \Rightarrow B}$	0 0 1 0	$x(1-y)$	$x-xy$
$f_5^{[2]}$	$\bar{B}$	1 0 1 0	$(1-x)(1-y)+xy$	$1-y$
$f_6^{[2]}$	XOR , $A \oplus B$	0 1 1 0	$(1-x)y+x(1-y)$	$x+y-2xy$
$f_7^{[2]}$	$NAND$, $\bar{A} \vee \bar{B}$	1 1 1 0	$(1-x)(1-y)+(1-x)y+x(1-y)$	$1-xy$
$f_8^{[2]}$	AND , $A \wedge B$	0 0 0 1	xy	xy
$f_9^{[2]}$	$A \Leftrightarrow B$	1 0 0 1	$(1-x)(1-y)+xy$	$1-x-y-2xy$
$f_{10}^{[2]}$	B	0 1 0 1	$(1-x)y+xy$	y
$f_{11}^{[2]}$	$A \Rightarrow B$	1 1 0 1	$(1-x)(1-y)+(1-x)y$	$1-x+xy$
$f_{12}^{[2]}$	A	0 0 1 1	$x(1-y)+xy+xy$	x
$f_{13}^{[2]}$	$A \Leftarrow B$	1 0 1 1	$(1-x)(1-y)+x(1-y)+xy$	$1-y+xy$
$f_{14}^{[2]}$	OR , $A \vee B$	0 1 1 1	$(1-x)y+x(1-y)+xy$	$x+y+xy$
$f_{15}^{[2]}$	T	1 1 1 1	$(1-x)(1-y)+(1-x)y+x(1-y)+xy$	1

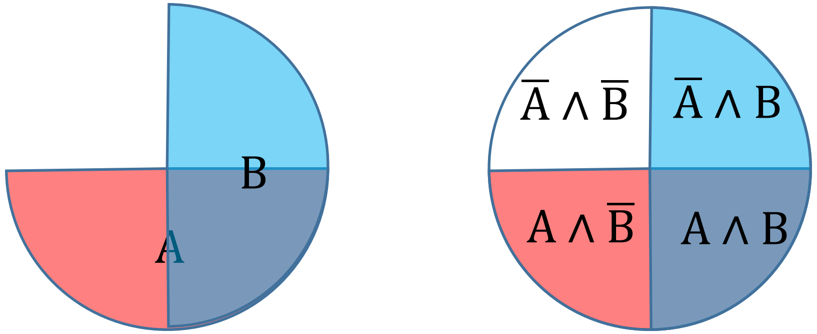
[*] Toffano, Z. **Eigenlogic in the Spirit of George Boole**. Logica Universalis, Birkhäuser-Springer, 14, 175–207 (2020).

Logical forms and diagrams

Elementary propositions: A **,** B

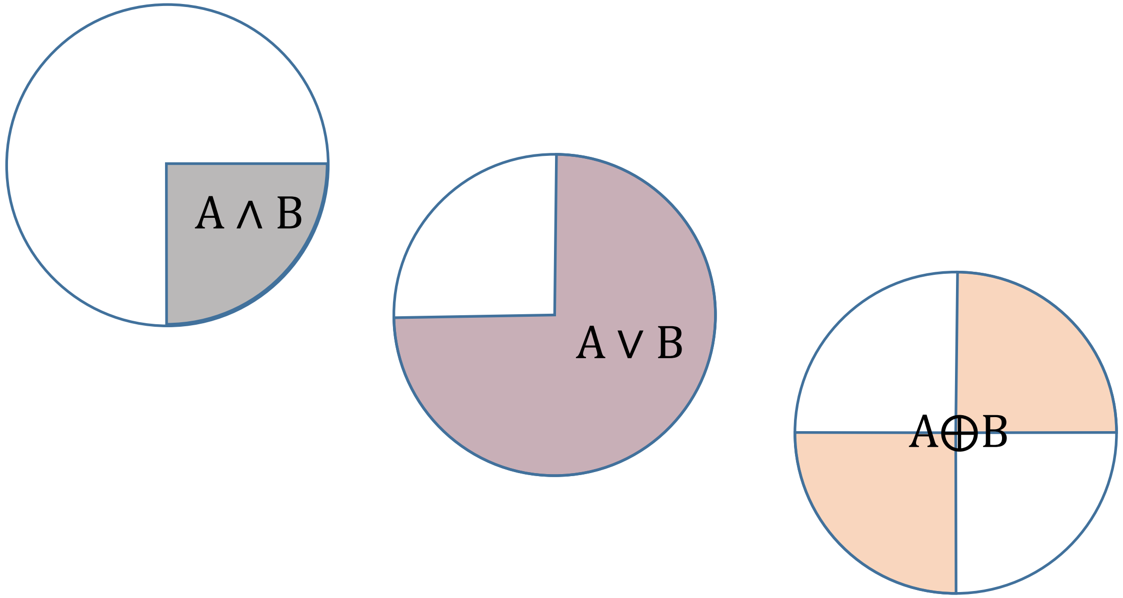


<u>**SOP** (Sum Of Products) canonical form</u>
disjunction (∨, OR) of conjunctions (∧, AND)
Conjunction: $\mathrm{A} \wedge \mathrm{B}$



Disjunction: $\mathrm{A} \vee \mathrm{B} = (\bar{\mathrm{A}} \wedge \mathrm{B}) \vee (\mathrm{A} \wedge \bar{\mathrm{B}}) \vee (\mathrm{A} \wedge \mathrm{B})$
in arithmetical form:
$$(1-a)b + a(1-b) + ab = a + b - ab$$

Exclusive disjunction: $\mathrm{A} \oplus \mathrm{B} = (\bar{\mathrm{A}} \wedge \mathrm{B}) \vee (\mathrm{A} \wedge \bar{\mathrm{B}})$
in arithmetical form:
$$(1-a)b + a(1-b) = a + b - 2ab$$



<u>**Reed-Muller** canonical form</u>
exclusive disjunction (⊕, XOR) of conjunctions (∧, AND)
$$\mathrm{A} \vee \mathrm{B} = (\mathrm{A} \oplus \mathrm{B}) \vee (\mathrm{A} \wedge \mathrm{B}) = \mathrm{A} \oplus \mathrm{B} \oplus (\mathrm{A} \wedge \mathrm{B})$$

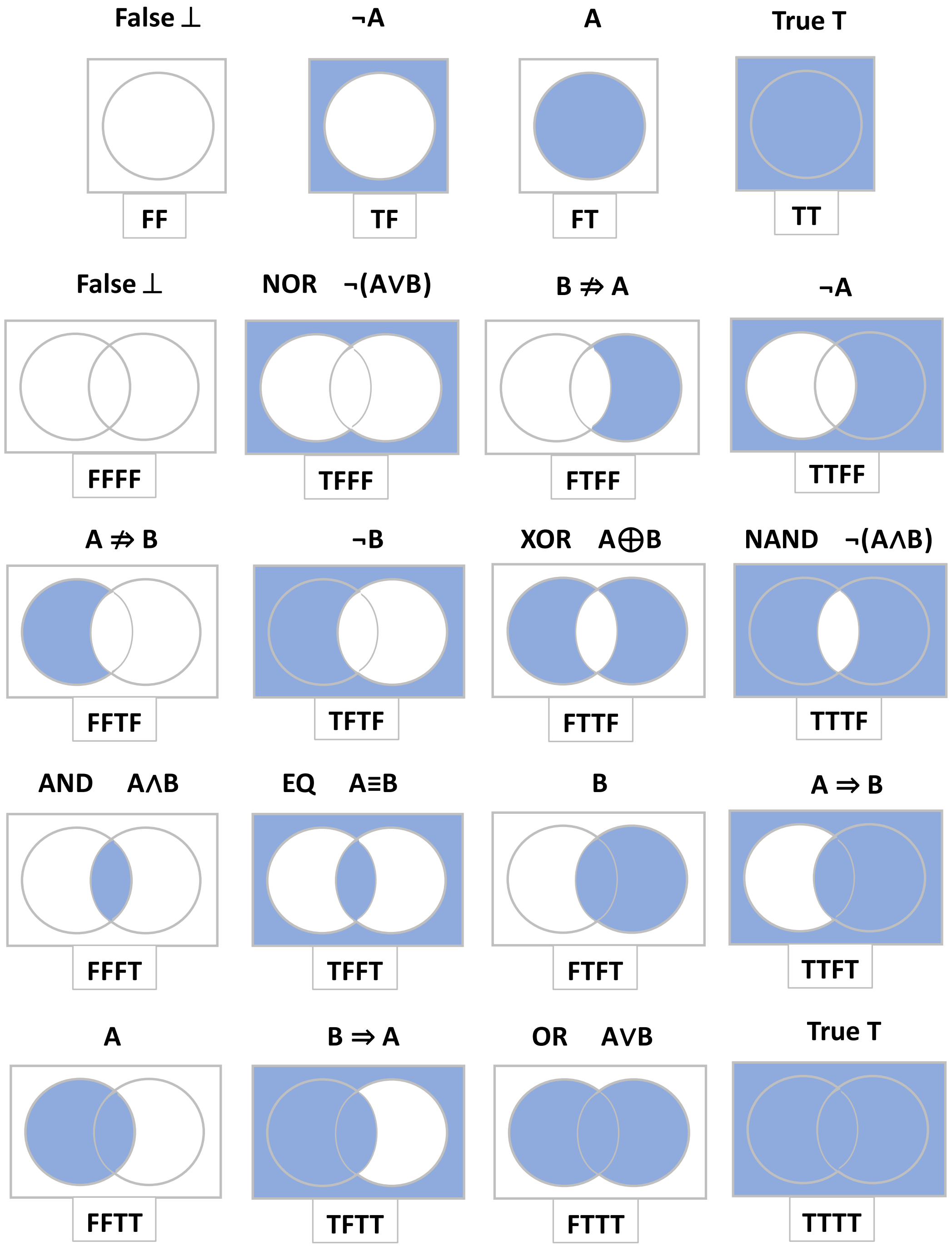
<u>**De Morgan** duality</u>: $\bar{\mathrm{A}} \wedge \bar{\mathrm{B}} = \overline{\mathrm{A} \vee \mathrm{B}}$
in arithmetical form:
$$(1-a)(1-b) = 1 - (a + b - ab) = 1 - a - b + ab$$

Truth tables and Venn diagrams

A simple way to illustrate all logical truth-tables is by using **Venn diagrams** (Venn 1881)

Direct correspondence with **set theory** and **probability theory**

Widely used in **information theory** for the representation of different information representations (relative, conditional…)



Operators in Logic

Why one should use operators (rather than functions) in logic ?

An important difference between operators and functions:

an operator can be described by its action only without defining the input domain

Operators permit to represent different effects peculiar to quantum physics:

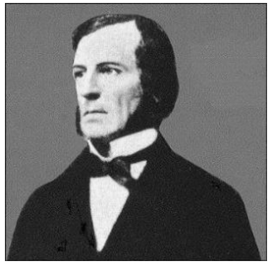
- non-commutativity
- reversibility

Quantum computing uses reversible operators (commuting and not) as logical quantum gates

Linear algebra using operators is the most used mathematical language to perform computer algorithms in the domains of:

- Big Data
- Artificial Intelligence (AI)
- Quantum Computing
- …

Operators in Logic



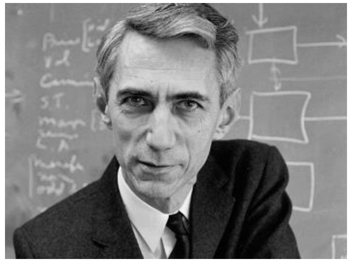
In 1847 **G. Boole** uses symbols (elective) that act as idempotent operators
less known: in 1848 in [a], he gave a logical interpretation of unitary **quaternions**.

C. S. Pierce often used matrices to build his logic formalism at the end of the XIXth century.

In 1921 **L. Wittgenstein** states in [b] that all propositions can be derived by repeated application of the operator $\boldsymbol{N}$ to the elementary propositions.


In 1924, **M. Schönfinkel** [c] introduced an operator based method in logic
H. Curry named it successively **Combinatory Logic** and improved the method [d] which lead to **lambda calculus**.

Boolean logical functions as used nowadays in digital circuits were essentially introduced by **C.E. Shannon** in his Master's thesis in 1938 [e] to represent binary switching functions.



[a] Boole, G. **Notes on quaternions**. Philos. Mag, 33, 278–280, (1848)

[b] Wittgenstein, L. **Tractatus Logico-Philosophicus**, 6.001, Routledge (1921)

[c] Schönfinkel, M. **Über die Bausteine der mathematischen Logik**. Math. Ann, 92, 305–316. (1924)

[d] Curry, H.B.; Feys, R. **Combinatory Logic**; North-Holland Co: Amsterdam, The Netherlands, (1958)

[e] Shannon, C.E. **A symbolic analysis of relay and switching circuits**. Trans. AIEE, 57(12), 713–723 (1938)

Projections as propositions and quantum logic

M. H. Stone gave the conditions for operations on projectors and commutativity [a] and established that each binary logical proposition corresponds by duality to the set of all its true valuations (**Stone Duality**).

J. von Neumann considered measurement projection operators as propositions in 1932 [b] and also stated that a quantum state $|\psi\rangle$ can be represented by a **density matrix** (rank-1 projection operator) :

$$\boldsymbol{\rho} = |\psi\rangle\langle\psi|$$

Quantum logic proposed by Garret Birkhoff and John von Neumann [c] suggested the replacement of Boolean algebras with the lattice of closed subspaces of a (finite) Hilbert space.

Quantum Logic had many promoters but also detractors.
<u>It is still not considered an "operational tool" for quantum computing</u>

[a] Marshall Harvey Stone: **Linear Transformations in Hilbert Space and Their Applications to Analysis**, p.70: "Projections", (1932)

[b] John von Neumann. **Mathematical Foundations of Quantum Mechanics**, Eng. Transl. (1955), p.249: "Projectors as Propositions", (1932)

[c] Garret Birkhoff, John von Neumann, **The Logic of Quantum Mechanics**. The Annals of Mathematics, 2nd Ser., 37 (4), 823-843 (1936)

Eigenlogic

Eigenlogic

Eigenlogic: a logical method using operators in linear algebra **[a,b,c]**

logical operators ⟺ logical connectives (syntax)

eigenvalues of logical operators ⟺ truth values (semantics)

eigenvectors of logical operators ⟺ interpretations (propositional cases)

Eigenlogic uses the Kronecker product to scale-up to more logical arguments (arity).

A single seed operator generates the entire logic.

[a] Dubois, F., Toffano, Z., **Eigenlogic: A Quantum View for Multiple-Valued and Fuzzy Systems**, QI 2016. In LNCS; Springer: Berlin; Vol 10106, (2017).

[b] Toffano, Z., **Eigenlogic in the Spirit of George Boole**. Logica Universalis, Birkhäuser-Springer, 14, 175–207 (2020)

[c] Toffano Z, Dubois F., **Adapting Logic to Physics: The Quantum-Like Eigenlogic Program**. Entropy. ; 22(2):139. (2020)

Eigenlogic: one-qubit Boolean logical operators

The qubits $|1\rangle$ and $|0\rangle$ define the **computational basis** (the "z" base): $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
eigenvectors of the **Pauli matrix** $\boldsymbol{\sigma}_z = \mathrm{diag}(+1, -1)$

Choice of the **logical seed projector** $\boldsymbol{\Pi} = |1\rangle\langle 1|$ (density matrix of qubit $|1\rangle$)

$\boldsymbol{\Pi}|1\rangle = |1\rangle\langle 1||1\rangle = |1\rangle = 1|1\rangle$, eigenvalue: 1 ; $\boldsymbol{\Pi}|0\rangle = |1\rangle\langle 1||0\rangle = 0 = 0\,|0\rangle$, eigenvalue: 0.

Logical operators as a linear development (equivalent to Boole's method):

$$\boldsymbol{F} = f(0)(\mathbb{I} - \boldsymbol{\Pi}) + f(1)\boldsymbol{\Pi} = \begin{pmatrix} f(0) & 0 \\ 0 & f(1) \end{pmatrix} = \mathrm{diag}(f(0), f(1))$$

the cofactors $f(0)$ and $f(1)$ are the eigenvalues *i.e.* the truth values of the logical connective.

Negation is obtained by complementation (substracting from the identity operator) : $\overline{\boldsymbol{F}} = \mathbb{I} - \boldsymbol{F}$

other choices of logical bases are possible: e.g. the "x" base with $\boldsymbol{\Pi}_+ = |+\rangle\langle +|$, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Eigenlogic: two-qubit Boolean logical operators

Making use of the Kronecker product ⊗ to scale up to more arguments (as done in quantum computing)

Scaling to 2-qubit logical operators with the 4 basis projectors:

$$\boldsymbol{\Pi}_{11} = |11\rangle\langle 11| = \boldsymbol{\Pi}\otimes\boldsymbol{\Pi}\ ;\ \boldsymbol{\Pi}_{10} = \boldsymbol{\Pi}\otimes(\mathbb{I}-\boldsymbol{\Pi})\ ;\ \boldsymbol{\Pi}_{01} = (\mathbb{I}-\boldsymbol{\Pi})\otimes\boldsymbol{\Pi}\ ;\ \boldsymbol{\Pi}_{00} = (\mathbb{I}-\boldsymbol{\Pi})\otimes(\mathbb{I}-\boldsymbol{\Pi})$$

All 16 logical operators can directly be obtained by the bilinear development (G. Boole's method)

$$\boldsymbol{F} = f(0,0)\boldsymbol{\Pi}_{00} + f(0,1)\boldsymbol{\Pi}_{01} + f(1,0)\boldsymbol{\Pi}_{10} + f(1,1)\boldsymbol{\Pi}_{11} = diag(f(0,0), f(0,1), f(1,0), f(1,1))$$

the truth values are $f(x,y) \in \{0,1\}$

Eigenlogic elementary (or atomic) propositions and connectives

In **propositional logic** one defines the **elementary** (or **atomic**) **propositions** P and Q in a **well-formed-formula**.

In Eigenlogic atomic propositions are operators that are extensions of the seed projector $\boldsymbol{\Pi}$:

$$\boldsymbol{P} = \boldsymbol{\Pi} \otimes \mathbb{I} = \mathrm{diag}(0,0,1,1) \quad , \quad \boldsymbol{Q} = \mathbb{I} \otimes \boldsymbol{\Pi} = \mathrm{diag}(0,1,0,1)$$

directly from $\boldsymbol{P}$ and $\boldsymbol{Q}$ all other logical operators are derrived:

Conjunction (AND, ∧) $\boldsymbol{F}_{\mathrm{AND}} = \boldsymbol{F}_{\mathrm{P \wedge Q}} = \boldsymbol{P} \cdot \boldsymbol{Q} = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi} = \mathrm{diag}(0,0,0,1)$ is the product of $\boldsymbol{P}$ and $\boldsymbol{Q}$

Disjunction (OR, ∨) $\boldsymbol{F}_{\mathrm{OR}} = \boldsymbol{F}_{\mathrm{P \vee Q}} = \boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{P} \cdot \boldsymbol{Q} = \mathrm{diag}(0,1,1,1)$ is not the sum of $\boldsymbol{P}$ and $\boldsymbol{Q}$

Exclusive disjunction (XOR, ⊕) $\boldsymbol{F}_{\mathrm{XOR}} = \boldsymbol{P} + \boldsymbol{Q} - 2\boldsymbol{P} \cdot \boldsymbol{Q} = \mathrm{diag}(0,1,1,0)$ is not the sum of $\boldsymbol{P}$ and $\boldsymbol{Q}$

Material implication (⟹) $\boldsymbol{F}_{\Rightarrow} = \boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{P} \cdot \boldsymbol{Q} = \mathrm{diag}(1,1,0,1)$

Negation is simply obtained by subtracting from the identity operator $\mathbb{I}$:

$$\boldsymbol{F}_{\boldsymbol{NAND}} = \mathbb{I} - \boldsymbol{F}_{\boldsymbol{AND}} = \mathrm{diag}(1,1,1,0) \quad ; \quad \boldsymbol{F}_{\Leftrightarrow} = \mathbb{I} - \boldsymbol{F}_{\mathrm{XOR}} = \mathrm{diag}(1,0,0,1)$$

Changing the paradigm: using truth values {+1,-1}

$+1$ (spin up) $\leftrightarrow$ 0 : "False" ; -1 (spin down) $\leftrightarrow$ 1 : "True"

The linear **Householder Transform**: $\boldsymbol{G} = \mathbb{I} - 2\boldsymbol{F} = (-1)^{\boldsymbol{F}} = e^{i\pi \boldsymbol{F}} = e^{i\frac{\pi}{2}} e^{-i\frac{\pi}{2}\boldsymbol{G}}$

Is an isomorphism from projection operators $\boldsymbol{F}$ (eigenvalues $\{0,1\}$) to involution operators $\boldsymbol{G}$ (eigenvalues$\{+1,-1\}$) :

$\boldsymbol{G}$ and $\boldsymbol{F}$ have the same eigenvectors

Choice of the logical "seed" operator: **Pauli matrix**, $\boldsymbol{\sigma_z} = \boldsymbol{Z}$ the « Z » quantum gate (other directions possible: $\boldsymbol{\sigma_x}$…)

$$\boldsymbol{\sigma_z} = \boldsymbol{Z} = \mathrm{diag}(+1,-1) = (\mathbb{I} - \boldsymbol{\Pi}) - \boldsymbol{\Pi} = \mathbb{I} - 2\boldsymbol{\Pi} = (-1)^{\boldsymbol{\Pi}}$$

The equivalent of the elementary propositions $\boldsymbol{P}$ and $\boldsymbol{Q}$ in $\{0,1\}$ are the involution $\boldsymbol{U}$ and $\boldsymbol{V}$ in $\{+1,-1\}$

$$\boldsymbol{U} = \boldsymbol{\sigma_z} \otimes \mathbb{I} = \mathrm{diag}(+1,+1,-1,-1) \quad , \quad \boldsymbol{V} = \mathbb{I} \otimes \boldsymbol{\sigma_z} = diag(+1,-1,+1,-1)$$

Negation is simply obtained by multiplying by -1: $\overline{\boldsymbol{G}} = -\boldsymbol{G}$

Exclusive disjunction $\boldsymbol{G}_{\oplus} = \boldsymbol{U} \cdot \boldsymbol{V} = \mathrm{diag}(1,-1,-1,1)$ is the product of $\boldsymbol{U}$ and $\boldsymbol{V}$

This shows that even though the logic for {0,1} and {+1,-1} is the same, the mathematical operations are not !
This **binary reversible logic alphabet** is often used in **Ising models** and **neural networks.**

Logical Operator Truth Tables

logical connective for P, Q	truth table {F, T}: {0, 1} or { + 1, − 1}	{0, 1} projective logical operator	{ + 1, − 1} involitive logical operator
False F	F F F F	0	+ I
NOR	F F F T	I − P − Q + PQ	(1/2) (+I − U − V − UV)
P ⇍ Q	F F T F	Q − PQ	(1/2) (+I − U + V+UV)
¬P	F F T T	I − P	− U
P ⇏ Q	F T F F	P − PQ	(1/2) (+ I + U − V + UV)
¬Q	F T F T	I − Q	− V
XOR ; P⊕Q	F T T F	P + Q − 2 PQ	UV = Z⊗Z
NAND ; P↑Q	F T T T	I − PQ	(1/2) (− I − U − V + UV)
AND ; P∧Q	T F F F	PQ = Π⊗Π	(1/2) (+ I + U + V − UV)
P ≡ Q	T F F T	I − P − Q + 2 PQ	− UV
Q	T F T F	Q = I⊗Π	V = I⊗Z
P ⇒ Q	T F T T	I − P + PQ	(1/2) (− I − U + V − UV)
P	T T F F	P = Π⊗I	U = Z⊗I
P ⇐ Q	T T F T	I − Q + PQ	(1/2) (− I + U − V − UV)
OR ; P∨Q	T T T F	P + Q − PQ	(1/2) (− I + U + V + UV)
True T	T T T T	I	− I

Eigenlogic is different from quantum logic

In Eigenlogic the **extension** of the seed operator $\boldsymbol{\Pi}$ with the identity operator $\mathbb{I}$

ensures the **independence** of the elementary propositions represented in Eigenlogic by the operators $\boldsymbol{P}$ and $\boldsymbol{Q}$

$$\boldsymbol{P} = \boldsymbol{\Pi} \otimes \mathbb{I} \quad ; \quad \boldsymbol{Q} = \mathbb{I} \otimes \boldsymbol{\Pi}$$

This is a major **difference with quantum logic** !

In quantum logic atomic propositions are **pure state density matrices** (rank-1 projection operators).

For example considering the density matrix of the quantum state $|11\rangle$:

$$\boldsymbol{\rho}_{11} = |11\rangle\langle 11| = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi}$$

corresponding to the **conjunction**

$$\boldsymbol{\rho}_{11} = \boldsymbol{F}_{\text{AND}} = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi} = \boldsymbol{P} \cdot \boldsymbol{Q}$$

which is not logically in propositional logic.

Many-Valued

More than binary: many-valued logic

Many-valued logics was first proposed by **J. Łukasiewicz [a]** and **E. Post [b]** in 1921

born nearly simultaneously to the theory of quantum mechanics.

With many-valued logic higher information densities can be achieved:
the information density in a m-valued system is $\log_2 m$ times larger than in a binary system

This logic has interested engineers involved in various aspects of information technology for over 40 years.

Used in the computer language HDL (Hardware Description Languages) for simulation of digital circuits and their synthesis.

Standards have been established, for example IEEE 1364MVL :

SYMBOL	MEANING
0	Logic zero
1	Logic one
Z	High-impedance state
X	Unknown logic value

[a] Jan Łukasiewicz, Selected Works, North-Holland, (1970), pp. 87–88, **On three-valued logic**, (1921)

[b] Emil Post, **Introduction to a General theory of Elementary Propositions**, American Journal of Mathematics 43: 163–185 (1921)

Many-valued Eigenlogic

The total number of logical connectives for a system of m values and n arguments is m^{m^n}.

For one-argument system with 3 values $3^{3^1} = 27$ and for two arguments $3^{3^2} = 19683$.

The seed operator $\boldsymbol{\Lambda}$ can be any operator with m non-degenerate eigenvalues λ_i,

using **Lagrange-Cayley-Hamilton matrix interpolation** the projector of each eigenstate is given uniquely by:

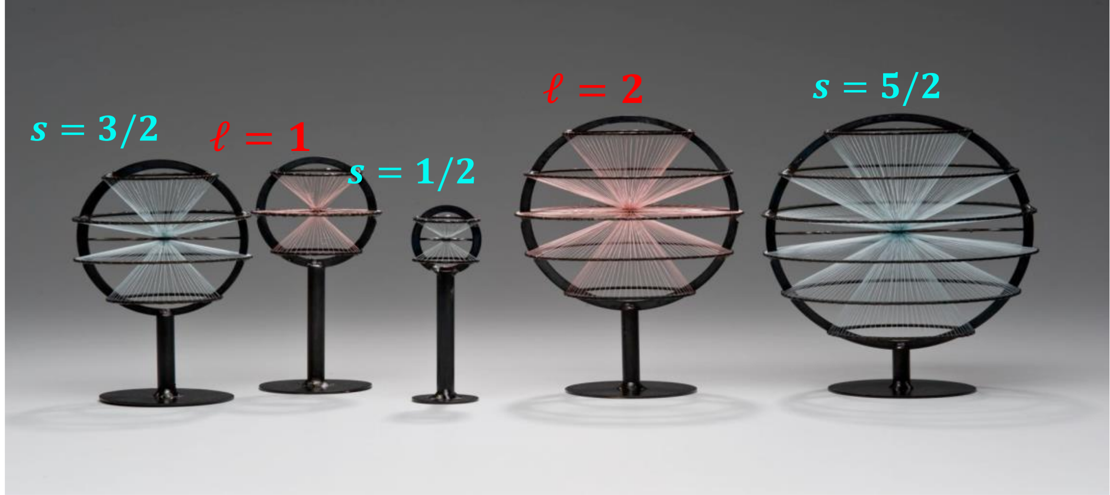
$$|\lambda_i >< \lambda_i| = \boldsymbol{\Pi}_{\lambda_i}(\boldsymbol{\Lambda}) = \prod_{j=1, j\neq i}^{m} \frac{\boldsymbol{\Lambda} - \lambda_j \mathbb{I}}{\lambda_i - \lambda_j}$$

A logical operator for arity-1 is then given by the spectral decomposition :

$$\boldsymbol{F}_{L} = \sum_{i=1}^{m} f(\lambda_j) \boldsymbol{\Pi}_{\lambda_i}(\boldsymbol{\Lambda})$$

Identification with Angular Momentum

Logical observables can be identified with
Quantum Angular Momentum.



Spin Family (Bosons ℓ and Fermions s)
(© Julian Voss-Andreae. Photo: Dan Kvitka.)

<u>Balanced ternary logic</u> equivalent to ***Orbital Angular Momentum*** (***OAM***) with $\ell = 1$.
The z component of the orbital angular momentum :

$$\boldsymbol{L_z} = \hbar \boldsymbol{\Lambda} = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = diag(+1,0,-1)$$

the three rank-1 projectors:

$$\boldsymbol{\Pi}_{+1} = \tfrac{1}{2}\boldsymbol{\Lambda}(\boldsymbol{\Lambda} + \mathbb{I}) \quad , \quad \boldsymbol{\Pi}_0 = \mathbb{I} - \boldsymbol{\Lambda}^2 \quad , \quad \boldsymbol{\Pi}_{-1} = \tfrac{1}{2}\boldsymbol{\Lambda}(\boldsymbol{\Lambda} - \mathbb{I})$$

$\boldsymbol{U}$ and $\boldsymbol{V}$, are then as usual in Eigenlogic:
$\boldsymbol{U} = \boldsymbol{\Lambda} \otimes \mathbb{I}, \ \boldsymbol{V} = \mathbb{I} \otimes \boldsymbol{\Lambda}$

Min U \\ V	F	N	T
False ≡ +1	+1	+1	+1
Neutral ≡ 0	+1	0	0
True ≡ − 1	+1	0	− 1

Max U \\ V	F	N	T
False ≡ +1	+1	0	− 1
Neutral ≡ 0	0	0	− 1
True ≡ − 1	− 1	− 1	− 1

In many-valued logic the Min and Max are the equivalent of AND and OR

$$\boldsymbol{Min}(\boldsymbol{U}, \boldsymbol{V}) = \frac{1}{2}(\boldsymbol{U} + \boldsymbol{V} + \boldsymbol{U}^2 + \boldsymbol{V}^2 - \boldsymbol{U}\cdot\boldsymbol{V} - \boldsymbol{U}^2\cdot\boldsymbol{V}^2) = diag(1,1,1,1,0,0, 1,0,-1)$$

$$\boldsymbol{Max}(\boldsymbol{U}, \boldsymbol{V}) = \frac{1}{2}(\boldsymbol{U} + \boldsymbol{V} - \boldsymbol{U}^2 - \boldsymbol{V}^2 + \boldsymbol{U}\cdot\boldsymbol{V} + \boldsymbol{U}^2\cdot\boldsymbol{V}^2) = diag(1,0,-1,0,0,-1,-1,-1,-1)$$

Syntax and Semantics in Unitary Operators

Syntax-Semantics duality and anticommutativity

Considering the 2 eigenstates $|\pm z\rangle$ of $\boldsymbol{\sigma}_z$ with eignevalues ± 1

and using he **anti-commutativity** of the Pauli operators: $\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_z = -\boldsymbol{\sigma}_z \cdot \boldsymbol{\sigma}_x$

$\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_z|\pm z\rangle = (\pm 1)\boldsymbol{\sigma}_x|\pm z\rangle = -\boldsymbol{\sigma}_z \cdot \boldsymbol{\sigma}_x|\pm z\rangle$ gives $\boldsymbol{\sigma}_z(\boldsymbol{\sigma}_x|\pm z\rangle) = (\mp 1)(\boldsymbol{\sigma}_x|\pm z\rangle)$

So $\boldsymbol{\sigma}_x|\pm z >$ is an eigenstate of $\boldsymbol{\sigma}_z$ with eigenvalue ∓ 1 it represents the state $|\mp z\rangle$ the complement of $|\pm z\rangle$

Identifying $|+z\rangle$ with qubit $|0\rangle$ and $|-z\rangle$ with qubit $|1\rangle$ gives : $\boldsymbol{\sigma}_x|0\rangle = |1\rangle$ and $\boldsymbol{\sigma}_x|1\rangle = |0\rangle$

This operation corresponds to **logical binary negation**.

So for these operators the basic logical operation of binary negation is a consequence of anti-commutativity

In this very simple example using the Pauli matrices as Eigenlogic operators, one has simultaneously:

- a **semantic representation** by the eigenvalues of the diagonal Pauli matrix $\boldsymbol{\sigma}_z$
- a **syntactic representation** by a permutation operation represented by the Pauli matrix $\boldsymbol{\sigma}_x$.

Syntax and Semantics for many-valued operators

The quantum state logical complementation can be generalized for a d-dimensional multi-level system (**qudit**)

using the generalization of the Pauli operators given by the Weyl operators: $\boldsymbol{X}_d$ and $\boldsymbol{Z}_d$:

$$\boldsymbol{X}_d|j\rangle = |j+1\rangle \quad ; \quad \boldsymbol{Z}_d|j> = \omega_d^j|j> \text{ with } \omega_d = e^{i\frac{2\pi}{d}}$$

$\boldsymbol{X}_d$ and $\boldsymbol{Z}_d$ possess the same eignevalues and verify : $\boldsymbol{Z}_d \cdot \boldsymbol{X}_d = \omega_d\, \boldsymbol{X}_d \cdot \boldsymbol{Z}_d$

the action of the **shift operator** $\boldsymbol{X}_d$ on the state $|j>$, which is an eigenstate of $\boldsymbol{Z}_d$, gives the state $|j+1>$, so by applying successively this operator one can generate all the other states of the basis.

The semantics is here represented by the eigenvalues of $\boldsymbol{Z}_d$ the d^{th} roots of unity ω_d

The syntax is represented by $\boldsymbol{X}_d$ corresponding to a **many-valued negation** as formulated by E. Post.

The transformation from $\boldsymbol{Z}_d$ to $\boldsymbol{X}_d$ is the **Discrete Fourier Transform (DFT)** operator (**Quantum Fourier Transform**)

The Fourier Transform becomes the mediator between logical syntax and logical semantics for many-valued systems.

Fuzzy

Fuzzy Eigenlogic: when the logical input is not an eigenstate

In 1965 **L. Zadeh** [a] proposed fuzzy logic to describe partial truths.
Truth values can take values between 0 and 1.

Fuzzy logic is grounded on the theory of fuzzy sets. The relation between the theory of fuzzy sets and the probability theory has been debated for a long time.

The quantum principle of superposition of states finds a counterpart in the degree of membership to fuzzy sets: the mean value of an Eigenlogic projection operator $\boldsymbol{F}$ gives a **fuzzy measure** when the quantum state $|\psi\rangle$ is not an eigenstate of $\boldsymbol{F}$ (a **crisp measure** 0 or 1 for eigenstates)

The Eigenlogic **fuzzy membership function** is: $\mu = \langle\psi|\boldsymbol{F}|\psi\rangle$ with $0 \leq \mu \leq 1$

Fuzziness can be related to the probabilistic nature of quantum measurements (Born rule).
For a projective observable $\boldsymbol{P}$ measured on a quantum state $|\psi\rangle$ we have the probability (**Gleason's theorem** [b]):

$$p_{|\psi\rangle} = \langle\psi|\boldsymbol{P}|\psi\rangle = Tr(\boldsymbol{\rho} \cdot \boldsymbol{P}) \qquad \text{with} \qquad \boldsymbol{\rho} = |\psi\rangle\langle\psi| \quad \text{the “density matrix”.}$$

A projective observable corresponds to a logical projection operator in Eigenlogic.

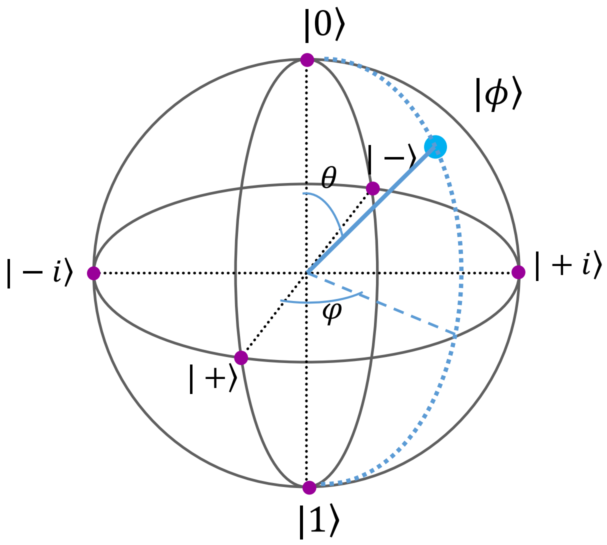
[a] Zadeh, L.A.: **Fuzzy sets**. Information and Control, 8 (3), 338-353, (1965)
[b] A. M. Gleason, **Measures on the closed subspaces of a Hilbert space**. Indiana U. Mathematics Journal, 6, 885–893, (1957)

Eigenlogic fuzzy conjunction, disjunction and material implication

A generic qubit state on the Bloch sphere: $|\phi\rangle = \sin\frac{\theta}{2}|0\rangle + e^{i\varphi}\cos\frac{\theta}{2}|1\rangle$,

the quantum average (Born rule) of the logical projector is: $\mu(A) = \langle\phi|\boldsymbol{\Pi}|\phi\rangle = \cos^2\frac{\theta}{2}$
and the complement: $\mu(\bar{A}) = \langle\phi|(\mathbb{I} - \boldsymbol{\Pi})|\phi\rangle = \sin^2\frac{\theta}{2} = 1 - \cos^2\frac{\theta}{2}$

which satisfies the condition of fuzzy logic for the complement (negation) of a fuzzy set.



Bloch sphere in Hilbert space

The corresponding fuzzy membership function for $\boldsymbol{P}$ and $\boldsymbol{Q}$ by performing the quantum average on : $|\psi\rangle = |\phi_p\rangle \otimes |\phi_q\rangle$ with $p = \left(\cos\frac{\theta_p}{2}\right)^2$ and $q = \left(\cos\frac{\theta_q}{2}\right)^2$

$$\mu(P) = \langle\psi|\boldsymbol{P}|\psi\rangle = p(1 - q) + p \cdot q = p \quad ; \quad \mu(Q) = \langle\psi|\boldsymbol{Q}|\psi\rangle = q$$

The fuzzy measure of logical operators are :

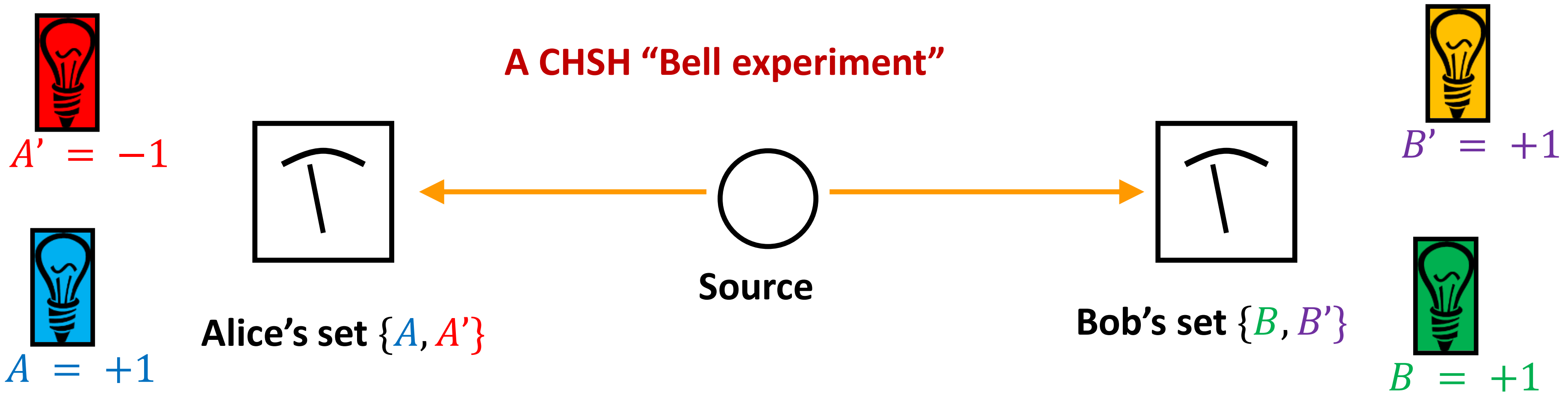
Conjunction $\mu(P \wedge Q) = \langle\psi|\boldsymbol{P} \cdot \boldsymbol{Q}|\psi\rangle = \langle\psi|\boldsymbol{\Pi} \otimes \boldsymbol{\Pi}|\psi\rangle = p \cdot q = \mu(P) \cdot \mu(Q)$
Disjunction $\mu(P \vee Q) = p + q - p \cdot q = \mu(P) + \mu(Q) - \mu(P) \cdot \mu(Q)$
Material Implication $\mu(P \Rightarrow Q) = 1 + p + p \cdot q = 1 - \mu(Q) + \mu(P) \cdot \mu(Q)$

We see that fuzzy disjunction $\mu(P \vee Q)$ corresponds to the usual **inclusion-exclusion** expression for combined probabilities due to **H. Poincaré [*]**

[*] Poincaré, H. **Calcul des Probabilités**; Gauthier-Villars: Paris, France, 1912

Logic and Bell Inequalities

Correlations of 2 quantum particles: CHSH Bell-inequality [*]



[*] J.F. Clauser; M.A. Horne; A. Shimony; R.A. Holt, **Proposed experiment to test local hidden-variable theories**, Phys. Rev. Lett., 23 (15): 880–4, (1969)

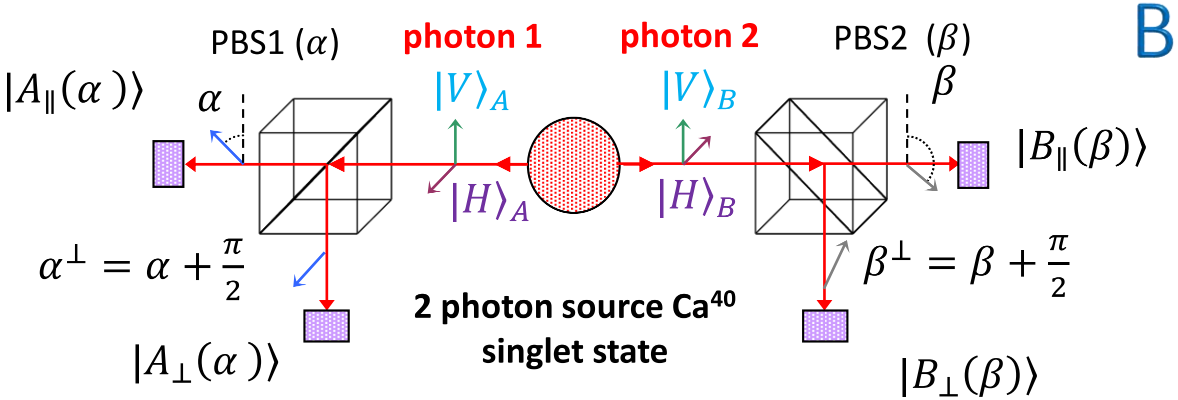
Local properties (*ex.* $A = +1, A' = -1, B = +1, B' = +1$)

in all 16 cases (deterministic) $\quad A(B + B') + A'(B - B') = \pm 2$

CHSH-Bell inequality requires 16 measurements: $\quad S = |< AB + AB' + A'B - A'B' >| \leq 2$

But Quantum Mechanics allows: $\mathbf{2 < S \leq 2\sqrt{2} = 2.83}$

so violates the CHSH Inequality > 2 !



Bell inequality: the Orsay experiment [*]

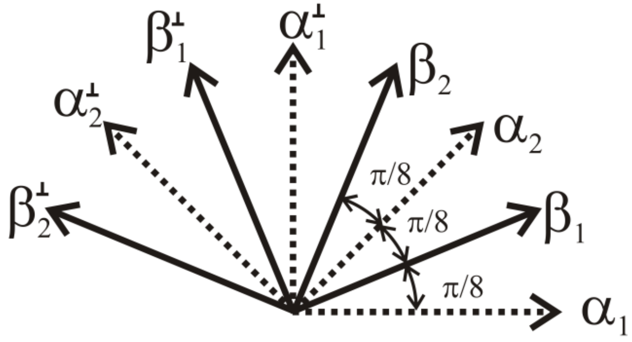
[*] A. Aspect; P. Grangier; G. Roger, **Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities**. Phys. Rev. Lett. 49 (2): 91–4, (1982)

$C(\alpha,\beta)$*:* coincidence photon counting rate function of the polarisation angles α and β

Then the <u>2-photon correlation</u> is defined: $$E(\alpha,\beta) = \frac{C(\alpha,\beta)+C(\alpha^{\perp},\beta^{\perp})-C(\alpha^{\perp},\beta)-C(\alpha,\beta^{\perp})}{C(\alpha,\beta)+C(\alpha^{\perp},\beta^{\perp})+C(\alpha^{\perp},\beta)+C(\alpha,\beta^{\perp})}$$

The **Bell CHSH inequality** is a function of the correlations E for 4 experimental settings $\{\alpha_i,\beta_j\}$ with $i,j \in \{1,2\}$:

the **Bell parameter**: $|S| = |E(\alpha_1,\beta_1) + E(\alpha_1,\beta_2) + E(\alpha_2,\beta_1) - E(\alpha_2,\beta_2)|$
Classically the Bell inequality verifies: $-2 \leq |S| \geq +2$



for the angles : $\beta_1 - \alpha_1 = \frac{\pi}{8}, \alpha_2 - \beta_1 = \frac{\pi}{8}, \beta_2 - \alpha_2 = \frac{\pi}{8}$

we have a <u>violation of the Bell CHSH inequality</u> $|S| \geq 2$

with the maximal value $|S|_{max} = 2\sqrt{2}$
obtained with an **entangled state (Bell state)** $|\Psi^{-}\rangle_s = \frac{1}{\sqrt{2}}(|H\rangle_1|V\rangle_2 - |V\rangle_1|H\rangle_2)$

The Bell CHSH inequality cases

Classical, local, separable

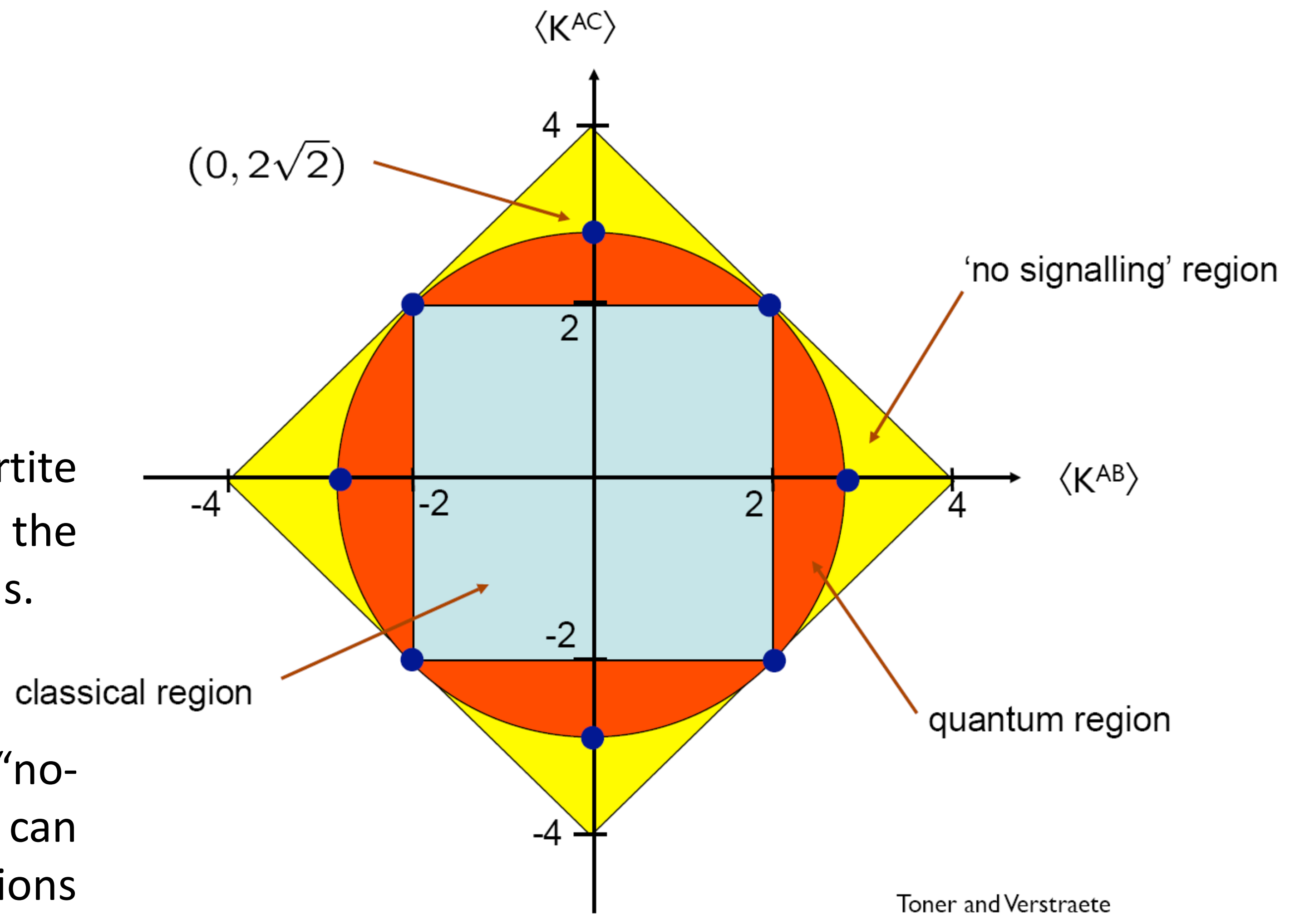
The Bell parameter S_{Bell} lies between 0 and 2.
Measurements are local: $E(X,Y) = E(X)E(Y)$.

Quantum

The case $2 \leq S_{Bell} \leq 2\sqrt{2}$ achieved with bipartite quantum entangled states. $S_{Bell} = 2\sqrt{2}$ is called the Tsirelson's bound and is a limit for Quantum systems.

No-signaling (post-quantum theories)

The case between $2\sqrt{2}$ and 4 is called the "no-signalling" region. The maximum value $S_{Bell} = 4$ can be attained with logical probabilistic constructions often named **non-local PR boxes**.



The CHSH Bell inequality and probabilities

The CHSH Bell inequality is expressed with the Pauli spin operators $\boldsymbol{\sigma}_i$ along the 4 measurement directions

The CHSH measurement operator is then : $\boldsymbol{CHSH} = \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B + \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_{B'} + \boldsymbol{\sigma}_{A'} \otimes \boldsymbol{\sigma}_B - \boldsymbol{\sigma}_{A'} \otimes \boldsymbol{\sigma}_{B'}$

Considering the projection operators, each term transforms as: $\boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B = (\mathbb{I} - 2\boldsymbol{\Pi}_A) \otimes (\mathbb{I} - 2\boldsymbol{\Pi}_B)$

replacing and simplifying :

$$\boldsymbol{CHSH} = 2\mathbb{I} - 4\boldsymbol{\Pi}_A \otimes \mathbb{I} - 4\mathbb{I} \otimes \boldsymbol{\Pi}_B + 4\boldsymbol{\Pi}_A \otimes \boldsymbol{\Pi}_B + 4\boldsymbol{\Pi}_A \otimes \boldsymbol{\Pi}_{B'} + 4\boldsymbol{\Pi}_{A'} \otimes \boldsymbol{\Pi}_B - 4\boldsymbol{\Pi}_{A'} \otimes \boldsymbol{\Pi}_{B'}$$

In this expression one recognizes the logical projection and conjunction operators

To evaluate the inequality one averages this operator: $\langle \psi | \boldsymbol{CHSH} | \psi \rangle$

By averaging the operator $\boldsymbol{\mathcal{F}} = \frac{1}{4}\boldsymbol{CHSH} - \frac{\mathbb{I}}{2}$ one obtains the **Fine inequality** for probabilities:

$\mathcal{F} = P(\mathrm{A} \wedge \mathrm{B}) + P(\mathrm{A} \wedge \mathrm{B}') + P(\mathrm{A}' \wedge \mathrm{B}) - P(\mathrm{A}' \wedge \mathrm{B}') - P(\mathrm{A}) - P(\mathrm{A})$ classically $-1 \leq \mathcal{F} \leq 0$

(equivalent to the classical CHSH BI $-2 \leq CHSH \leq +2$) for entangled states one has violation of these inequalities.

George Boole already discussed these probability inequalities in 1854 as stated bay Itamar Pitowsky in **[*]**

[*] Pitowsky I. **From George Boole To John Bell — The Origins of Bell's Inequality**. In: Kafatos M. (eds) Bell's Theorem, Quantum Theory and Conceptions of the Universe. Fundamental Theories of Physics, vol 37. Springer (1989)

The strange propositions of Diederik Aerts [*]

D. Aerts in 1982 proposed a **macroscopic experiment** that violates the CHSH Bell Inequality maximally.

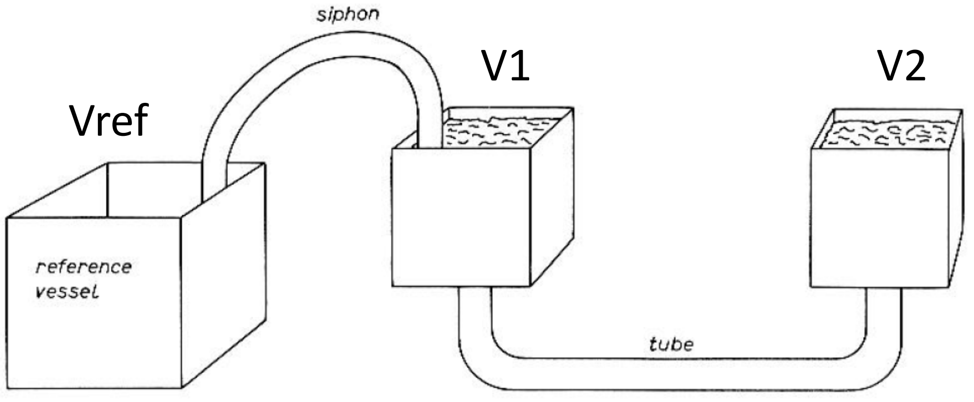


Fig. 1.

Two vessels V1 and V2 with a capacity of 8 liters each, linked through a tube with a capacity of 16 liters (at most the system holds 32 liters).
The vessel Vref used to siphon water from the V1 and V2 basins.

[*] D. Aerts, **Example of a macroscopical situation that violates Bell inequalities**, Lett. Nuovo Cimento **34**, 107 (1982)

<u>4 experiments</u> :

Experiment α: answer to: *is Vref > 10 L ?*
Experiment β: answer to: *are V1 or V2 > 6L ?*
Experiment γ: answer to: *is the water drinkable ?*
Experiment δ: answer to: *is the water transparent ?*

<u>outcomes: +1 if the answer is **YES** and −1 if the answer is **NO**</u>

Results of correlated experiments:

$X_{\alpha,\beta} = -1$, $X_{\alpha,\gamma} = +1$, $X_{\delta,\beta} = +1$ and $X_{\delta,\gamma} = +1$.

Taking the sum for the CHSH Bell parameter:

$$S = |X_{\alpha,\beta} - X_{\alpha,\gamma}| + |X_{\delta,\beta} + X_{\delta,\gamma}| = 4$$

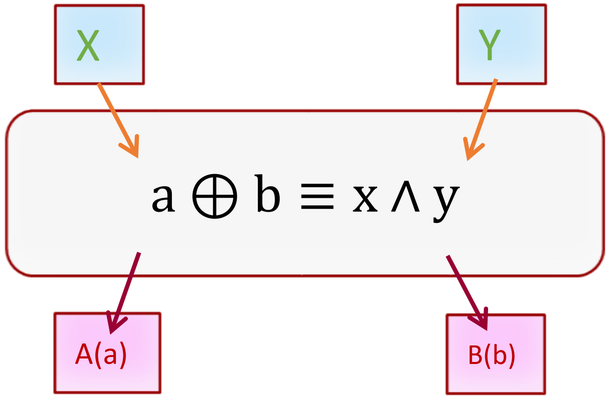
Bell's inequality is therefore maximally violated!

In 2012 we undertook the experiment in Supélec using 2 flower pots and a 32 m water tube. students: Vincent DUMOULIN & Yves SOURRILLE

The PR Box [*]

The well known nonlocal PR box correlates outputs (a, b) to inputs (x, y) in a two-party correlation by means of a logical constraint equation:

$$a \oplus b \equiv x \wedge y$$



This box violates the CHSH Bell Inequality (BI) maximally
The measurement outcomes (A, B) , Alice and Bob, give the values ± 1.

We define the joint mean value for the possible outcomes of the box as a function of the marginal probabilities [3]:

$$C_{x,y} = \sum_{a,b} P(a, b|a \oplus b \equiv x \wedge y) \cdot A(a) \cdot B(b)$$

where $A(a) = 1 - 2a = (-1)^a$; $B(b) = 1 - 2b = (-1)^b$

The Bell parameter considering the four input possibilities is:

$$S = C_{00} + C_{01} + C_{10} - C_{11} = 4$$

x	0	0	1	1
y	0	1	0	1
x ∧ y	0	0	0	1
C_{xy}	+1	+1	+1	-1
a ⊕ b	0	1	1	0
a , A	0 +1	0 +1	1 -1	1 -1
b , B	0 +1	1 -1	0 +1	1 -1

$$C_{xy} = P(0,0|x,y)\cdot(+1)\cdot(+1) + P(0,1|x,y)\cdot(+1)\cdot(-1) + P(1,0|x,y)\cdot(-1)\cdot(+1) + P(1,1|x,y)\cdot(-1)\cdot(-1)$$

$$C_{00} = C_{01} = C_{10} = \frac{1}{2} + 0 + 0 + \frac{1}{2} = +1$$

$$C_{11} = 0 - \frac{1}{2} - \frac{1}{2} + 0 = -1$$

[*] Popescu, S., Rohrlich, D. **Quantum nonlocality as an axiom**. Found Phys 24, 379–385 (1994)

Analysing the PR Box Bell Inequality by Eigenlogic

One uses the logical expression directly in an operator form using the following logical identity:

$$\mathrm{a} \oplus \mathrm{b} \equiv \mathrm{x} \wedge \mathrm{y} \quad \leftrightarrow \quad \overline{\mathrm{a} \oplus \mathrm{b} \oplus \mathrm{x} \wedge \mathrm{y}}$$

Using the involution properties: $(-1)^{\overline{\mathrm{a}\oplus\mathrm{b}\oplus\mathrm{x}\wedge\mathrm{y}}} = -(-1)^{\mathrm{a}\oplus\mathrm{b}\oplus\mathrm{x}\wedge\mathrm{y}} = -(-1)^{\mathrm{a}}(-1)^{\mathrm{b}}(-1)^{\mathrm{x}\wedge\mathrm{y}}$

One can then express the operator $\boldsymbol{G}_{\mathrm{PR}}$ with eigenvalues (truth values) : $-(-1)^{a}(-1)^{b}(-1)^{xy}$

The corresponding projective operator is: $\boldsymbol{F}_{\mathrm{PR}} = \frac{1}{2}(\mathbb{I} - \boldsymbol{G}_{\mathrm{PR}})$

The BI value is obtained by averaging the operator $\boldsymbol{G}_{\mathrm{PR}}$ on the possible situations given by the logical constraint $\mathrm{a} \oplus \mathrm{b} \equiv \mathrm{x} \wedge \mathrm{y}$ that is to the 8 cases out of 16 where the truth value of $\boldsymbol{F}_{\mathrm{PR}}$ is 1

Considering all the possible cases for $C_{x,y}$ one gets the maximum Bell parameter:

using of the idempotence property: ${\boldsymbol{F}_{\mathrm{PR}}}^{2} = \boldsymbol{F}_{\mathrm{PR}}$

$$S = -\frac{8}{16}\mathrm{Tr}(\boldsymbol{F}_{\mathrm{PR}} \cdot \boldsymbol{G}_{\mathrm{PR}}) = -\frac{1}{2}\mathrm{Tr}\big(\boldsymbol{F}_{\mathrm{PR}}(\mathbb{I} - 2\boldsymbol{F}_{\mathrm{PR}})\big) = \frac{1}{2}\mathrm{Tr}(\boldsymbol{F}_{\mathrm{PR}}) = \frac{8}{2} = 4$$

Generalising the PR Box for all logical bipartite constraints

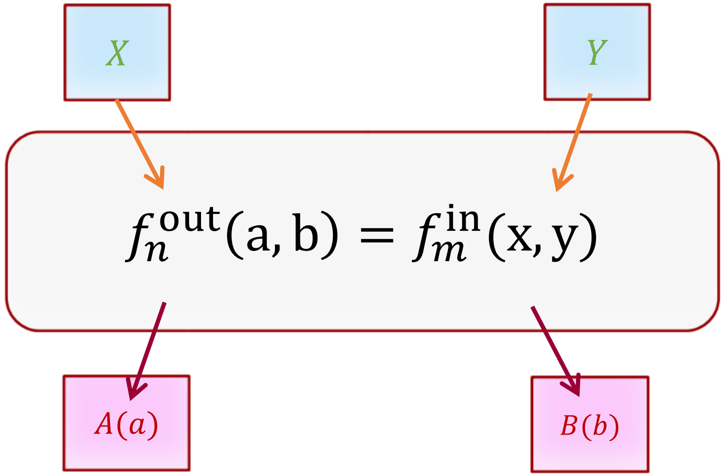
In total $16 \times 16 = 256$ logical equations.

The PR box corresponds to : $\mathrm{a} \oplus \mathrm{b} \equiv \mathrm{x} \wedge \mathrm{y}$ (in the table $f_6(\mathrm{a},\mathrm{b}) \equiv f_8(\mathrm{x},\mathrm{y})$)
BI violation $|S| = 4$ for other 15 no-signaling nonlocal boxes (**orange**)

Other 32 nonlocal boxes (**green**) are signaling and violate BI with

$$|S| = \frac{10}{3} \approx 3.33 > 2\sqrt{2} > 2$$

case for: $\mathrm{a} \vee \mathrm{b} \equiv \mathrm{x} \wedge \mathrm{y}$ (in the table $f_{14}(\mathrm{a},\mathrm{b}) \equiv f_8(\mathrm{x},\mathrm{y})$)
also exceeding the quantum limit.



Logical function for p, q $f_n(p,q)$	Truth table for 2 inputs (p,q) (1,1) (1,0) (0,1) (0,0)	Logical operator
f_0	(0 0 0 0)	contradiction : False ; ⊥
f_1	(0 0 0 1)	not or : NOR ; ¬P ∧ ¬Q
f_2	(0 0 1 0)	non-Implication : P ⊇ Q
f_3	(0 0 1 1)	negation of p : ¬P
f_4	(0 1 0 0)	converse non-implication : P ⊆ Q
f_5	(0 1 0 1)	negation of q : ¬Q
f_6	(0 1 1 0)	exclusive or : XOR ; P ⊕ Q
f_7	(0 1 1 1)	not and : NAND ; ¬P ∨ ¬Q
f_8	(1 0 0 0)	conjunction (and) : AND ; P ∧ Q
f_9	(1 0 0 1)	equivalence : XNOR ; P ≡ Q
f_{10}	(1 0 1 0)	right projection of q : Q
f_{11}	(1 0 1 1)	converse implication P ⊂ Q
f_{12}	(1 1 0 0)	left projection of p : P
f_{13}	(1 1 0 1)	implication : P ⊃ Q
f_{14}	(1 1 1 0)	disjunction (or) : OR ; P ∨ Q
f_{15}	(1 1 1 1)	tautology : True ; ⊤

OUTPUT / INPUT	f0	f1	f2	f4	f8	f3	f5	f12	f10	f6	f9	f7	f11	f13	f14	f15
f0																
f1										4	4	3,33	3,33	3,33	3,33	
f2										4	4	3,33	3,33	3,33	3,33	
f4										4	4	3,33	3,33	3,33	3,33	
f8										4	4	3,33	3,33	3,33	3,33	
f3																
f5																
f12																
f10																
f6																
f9																
f7		3,33	3,33	3,33	3,33					4	4					
f11		3,33	3,33	3,33	3,33					4	4					
f13		3,33	3,33	3,33	3,33					4	4					
f14		3,33	3,33	3,33	3,33					4	4					
f15																

Quantum Computing and Eigenlogic

Quantum Computing

A quantum computer is one whose operation exploits certain very special transformations of its internal state.

For computer scientists the most striking thing about quantum computation is that a quantum computer can be vastly more efficient than anything ever imagined in the classical theory of computational complexity, for certain computational tasks of considerable practical interest.

The time it takes the quantum computer to accomplish such tasks scales up much more slowly with the size of the input than it does in any classical computer.

Eigenlogic and quantum gates

Eigenlogic makes a correspondence between **quantum control logic** (David Deutsch's quantum logical gate paradigm) and ordinary propositional logic.
It is known that the 2-quibit **control-phase gate** $\boldsymbol{C_Z}$ in association with 1-quibit gates is a universal gate set.

In Eigenlogic the $\boldsymbol{C_Z}$ gate corresponds to the $\boldsymbol{AND}$ involution gate $\qquad \boldsymbol{C_Z} = \boldsymbol{G}_\wedge = diag(1,1,1,-1) = \boldsymbol{C_Z}$

The XOR gate (not universal) is given by the Kronecker product: $\qquad \boldsymbol{G}_\oplus = diag(1,-1,-1,1) = \boldsymbol{Z} \otimes \boldsymbol{Z}$

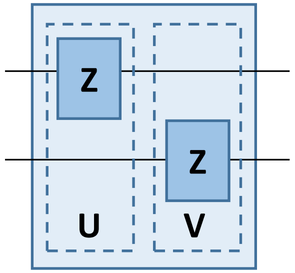
The well-known control-not **CNOT** *gate* $\boldsymbol{C}$ can be expressed using the Pauli matrices $\boldsymbol{\sigma_z} = \boldsymbol{Z}$ and $\boldsymbol{\sigma_x} = \boldsymbol{X}$

from the seed operators : $\boldsymbol{\Pi} = |1\rangle\langle 1| = \frac{1}{2}(\mathbb{I} - \boldsymbol{Z})$ and $\boldsymbol{\Pi}_+ = |+\rangle\langle +| = \frac{1}{2}(\mathbb{I} - \boldsymbol{X})$

using the **Eigenlogic involution conjunction operator** (in the alphabet $\{+1,-1\}$) :

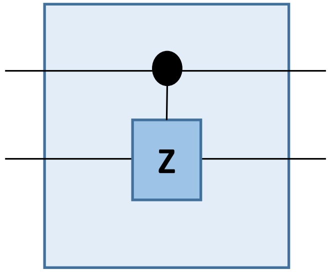
$$\boldsymbol{C} = (-1)^{\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}_+} = \mathbb{I} - 2(\boldsymbol{\Pi} \otimes \boldsymbol{\Pi}_+) = \frac{1}{2}(\mathbb{I} + \boldsymbol{Z} \otimes \mathbb{I} + \mathbb{I} \otimes \boldsymbol{X} - \boldsymbol{Z} \otimes \boldsymbol{X}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\boldsymbol{G}_\oplus = \boldsymbol{U}.\boldsymbol{V}$



Eigenlogic quantum gates

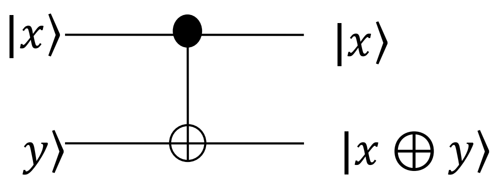
$\boldsymbol{C_Z} = \boldsymbol{G}_\wedge = ½(\boldsymbol{I} + \boldsymbol{U} + \boldsymbol{V} - \boldsymbol{U}.\boldsymbol{V})$



$\mathbf{CNOT} = \boldsymbol{C} = ½(\boldsymbol{I} + \boldsymbol{U} + \boldsymbol{V}_x - \boldsymbol{U}.\boldsymbol{V}_x)$

control bit

target bit



Control gates and quantum entanglement concurrence

For a general 2-qubit state: $|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$

Simple way to characterize entanglement by the **quantum concurrence**: $\mathrm{C} = 2|\alpha\delta - \beta\gamma|$

$\mathrm{C} = 0$ not entangled ; $0 < \mathrm{C} \leq 0$ entangled ; $\mathrm{C} = 1$ fully entangled

Entangling gate : the control-phase gate $\boldsymbol{C_Z} = \boldsymbol{G}_\wedge$

using $|++\rangle = |+\rangle \otimes |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$

the concurrence of $|++\rangle$ is: $2\left|\frac{1}{2}\frac{1}{2} - \frac{1}{2}\frac{1}{2}\right| = 0$ the state is not entangled

Let's apply $\boldsymbol{C_Z}|++\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) = \frac{1}{\sqrt{2}}(|0+\rangle - |1-\rangle)$

the resulting state is one of the Bell states in the basis $|z, x\rangle)$
called also a **cluster state** used in **measurement based quantum computing**

The concurrence $\boldsymbol{C_Z}|++\rangle$ is: $2\left|\frac{1}{2}\frac{-1}{2} - \frac{1}{2}\frac{1}{\sqrt{2}}\right| = 1$ the state is fully entangled

logical universality and quantum entanglement

In binary propositional logic the following set when combined with negation (NOT) constitutes

a **universal set of 8 logical connectives**: AND, OR, NOR, NAND, ⇒ , ⇏, ⇐ , ⇍
<u>one observes that they possess an **odd number of True and False truth values**</u>

the other 8 connectives are not: P, Q, ¬P, ¬Q, ≡ , XOR, F ,T
<u>one observes that they possess an **even number of True and False truth values**</u>

For involution logical operators $\boldsymbol{G}$ with eigenvalues {+1,-1}
the universal logical gates correspond to 8 operators with an odd number of eigenvalues $+1$ and -1
these are all entangling gates
the 8 other logical operators are separable (not netngled) and not universal.

This states clearly the <u>correspondence between logical universality and entanglement</u>.

P	Q	F	NOR	P ⇍ Q	¬P	P ⇏ Q	¬Q	XOR	NAND	AND	P ≡ Q	Q	P ⇒ Q	P	Q ⇒ P	OR	T
+	+	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
+	−	+	+	−	−	−	+	−	−	+	+	−	−	+	+	−	−
−	+	+	+	+	+	+	−	−	−	+	+	+	+	−	−	−	−
−	−	+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−

building the 3-qubit Toffoli universal quantum gate

The 3-qubit Toffoli (double-CNOT) gate $\boldsymbol{TO}$ is a **universal** reversible logic quantum gate directly in Eigenlogic form

$$\boldsymbol{TO} = \mathbb{I} - 2(\boldsymbol{\Pi}_Z \otimes \boldsymbol{\Pi}_Z \otimes \boldsymbol{\Pi}_X) = \frac{1}{4}(3\,\mathbb{I} + \boldsymbol{Z}_2 + \boldsymbol{Z}_1 + \boldsymbol{X}_0 - \boldsymbol{Z}_2 \cdot \boldsymbol{Z}_1 - \boldsymbol{Z}_2 \cdot \boldsymbol{X}_0 - \boldsymbol{Z}_1 \cdot \boldsymbol{X}_0 + \boldsymbol{Z}_2 \cdot \boldsymbol{Z}_1 \cdot \boldsymbol{X}_0)$$

Can be put in exponential form using the Householder transform

$$\boldsymbol{TO} = e^{+i\frac{\pi}{8}} e^{-i\frac{\pi}{8}\boldsymbol{Z}_1} e^{-i\frac{\pi}{8}\boldsymbol{Z}_2} e^{-i\frac{\pi}{8}\boldsymbol{X}_0} e^{i\frac{\pi}{8}\boldsymbol{Z}_2\cdot\boldsymbol{Z}_1} e^{i\frac{\pi}{8}\boldsymbol{Z}_2\cdot\boldsymbol{X}_0} e^{i\frac{\pi}{8}\cdot\boldsymbol{Z}_1\cdot\boldsymbol{X}_0} e^{-i\frac{\pi}{8}\boldsymbol{Z}_2\cdot\boldsymbol{Z}_1\cdot\boldsymbol{X}_0}$$

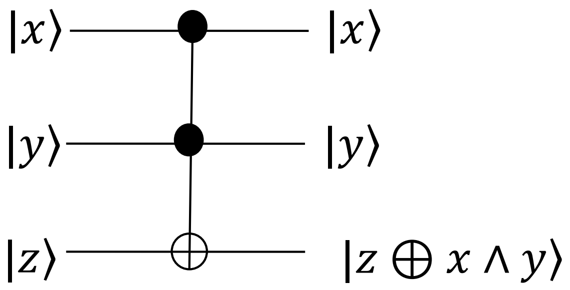
Alternative method using **T gates** [*]: $\boldsymbol{T} = \boldsymbol{Z}^{\frac{1}{4}} = e^{i\frac{\pi}{8}} e^{-i\frac{\pi}{8}\boldsymbol{Z}} = \mathrm{diag}\left(1, e^{i\frac{\pi}{4}}\right)$

using the **Reed-Muller form**: $\boldsymbol{C}_{\boldsymbol{CZ}} = \boldsymbol{T}_0 \cdot \boldsymbol{T}_1 \cdot \boldsymbol{T}_2 \cdot (\boldsymbol{T}_{x\oplus y})^{\dagger} \cdot (\boldsymbol{T}_{x\oplus z})^{\dagger} \cdot (\boldsymbol{T}_{y\oplus z})^{\dagger} \cdot (\boldsymbol{T}_{x\oplus y\oplus z})$

using the **Hadamard gate** extension one has again the Toffoli gate : $\boldsymbol{TO} = \boldsymbol{H}_0 \cdot \boldsymbol{C}_{\boldsymbol{CZ}} \cdot \boldsymbol{H}_0$

[*] Selinger, P., **Quantum circuits of T-depth one**, Phys. Rev. A, 87, 252–259, (2013)

control bit



$$\boldsymbol{TO} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Deutsch algorithm [*]

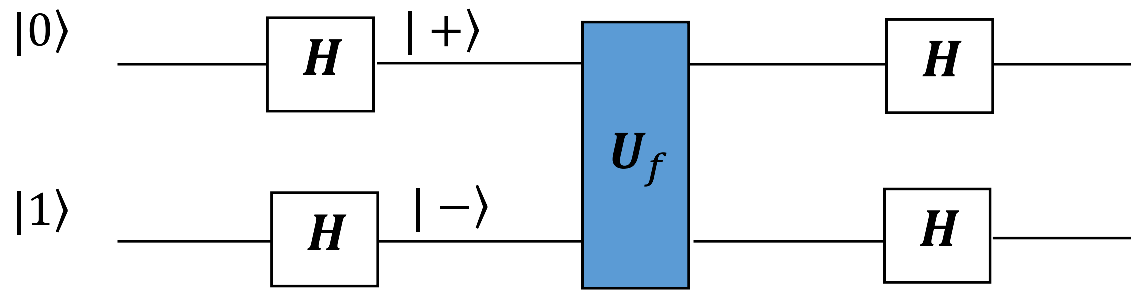
The Deutsch algorithm is <u>one of the first quantum algorithms more efficient than its classical counterpart</u>.

> The answer to the question: is the <u>logical function $f(x)$</u> constant or balanced?
> can be performed by a quantum computer in one step.
> (the classic treatment requires two steps.)

x	$f_{00}(x)$	$f_{01}(x)$	$f_{10}(x)$	$f_{11}(x)$
0	0	0	1	1
1	0	1	0	1
	constant	balanced	balanced	constant

[*] D. Deutsch, **Quantum theory, the Church-Turing principle and the universal quantum computer**, Proc. R. Soc. A, 400, 97–117, (1985)

The algorithm measurement is made on the upper qubit of the following circuit

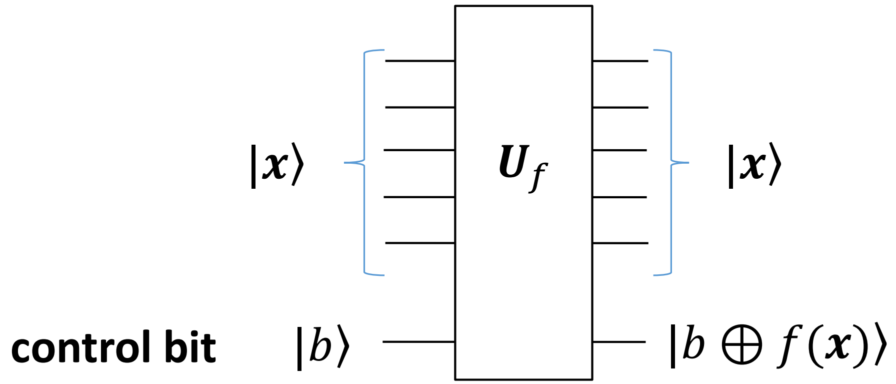


$$= (-1)^{f(0)}\,|f(0)\oplus f(1)\rangle|1\rangle$$
$$= (-1)^{f(0)}|0\rangle|1\rangle \text{ if constant}$$
$$= (-1)^{f(0)}|1\rangle|1\rangle \text{ if balanced}$$

$$\boldsymbol{U_f}\colon |x\rangle|y\rangle \to |x\rangle|y\oplus f(x)\rangle$$

Expressing the quantum oracle circuit in Eigenlogic

In quantum control logic Boolean logical functions can be calculated by means of a **quantum oracle** $\boldsymbol{U_f}$:



Particular cases are:
the 2-qubit CNOT $\boldsymbol{C}$
the 3-qubit Toffoli gate $\boldsymbol{TO}$ (f is the AND function)

The logical function f is represented by the projective Eignelogic operator $\boldsymbol{F}$

The control bit corresponds to the seed projection operator $\boldsymbol{\Pi}_+ = |+\rangle\langle+| = \frac{1}{2}(\mathbb{I} - \boldsymbol{X})$ (in the $|x\rangle$ basis).

The oracle is then simply expressed in Eigenlogic as : $\boldsymbol{U_f} = (-1)^{\boldsymbol{F}\otimes\boldsymbol{\Pi}_+} = \mathbb{I} - 2\boldsymbol{F}\otimes\boldsymbol{\Pi}_+$

In the case of a one bit Boolean function $f(x)$: $\boldsymbol{F} = f(0)\boldsymbol{\Pi}_0 + f(1)\boldsymbol{\Pi}_1 = f(0)(\mathbb{I} - \boldsymbol{\Pi}) + f(1)$

by developing one obtains another expression for the oracle : $\boldsymbol{U_f} = \boldsymbol{\Pi}_0 \otimes \boldsymbol{X}^{f(0)} + \boldsymbol{\Pi}_1 \otimes \boldsymbol{X}^{f(1)}$

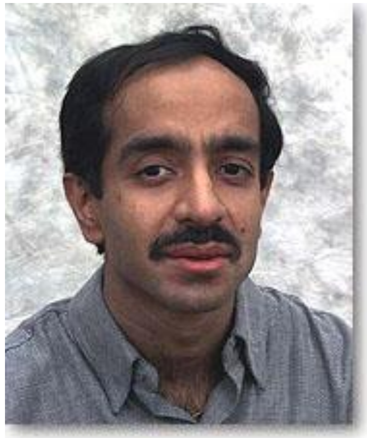
the **Deutsch algorithm** result obtained by applying the oracle on : $|+\rangle|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$\boldsymbol{U_f}|+\rangle|-\rangle = \frac{1}{\sqrt{2}}|0\rangle(-1)^{f(0)}|-\rangle + \frac{1}{\sqrt{2}}|1\rangle(-1)^{f(1)}|-\rangle$$

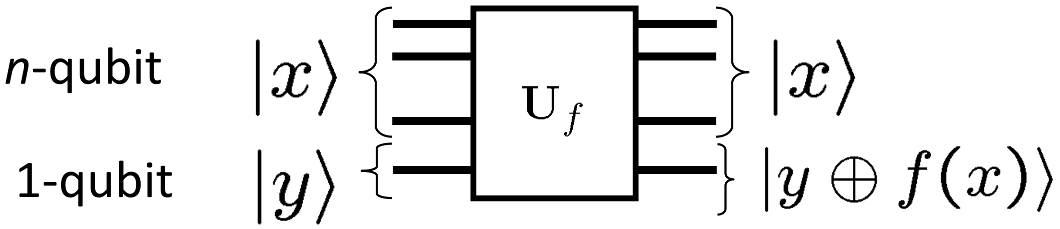
for constant $f(0) = f(1)$ $\quad \boldsymbol{U_f}|+\rangle|-\rangle = \pm|+\rangle|-\rangle$ and for balanced $f(0) \neq f(1)$ $\quad \boldsymbol{U_f}|+\rangle|-\rangle = \pm|-\rangle|-\rangle$

Grover's search algorithm [*]

Problem: finding the value x_0 in a large database with the fewest queries possible.



We consider a “black box” (oracle $\mathbf{U}_f$)
Having for the value x_0 the property :
$f(x_0) = 1$ and $f(x) = 0$ for $x \neq x_0$



Classical query:	complexity: $O(\exp(n))$
Grover quantum query:	complexity: $O(\exp(\sqrt{n}))$

[*] L. K. Grover. **A fast quantum mechanical algorithm for database search**, Proceedings, 28th Annual ACM Symp. on the Theory of Computing, p. 212, (1996)

Grover algorithm and first-order-logic

The **Grover search algorithm** looks for an element a (here $|a_2 a_1 a_0\rangle = |111\rangle$) satisfying the property P (**oracle**) <u>ca</u>
<u>be interpreted as an existential logical quantifier ∃</u>,
becomes the predicate proposition in first-order-logic: $\exists a\ P(a)$

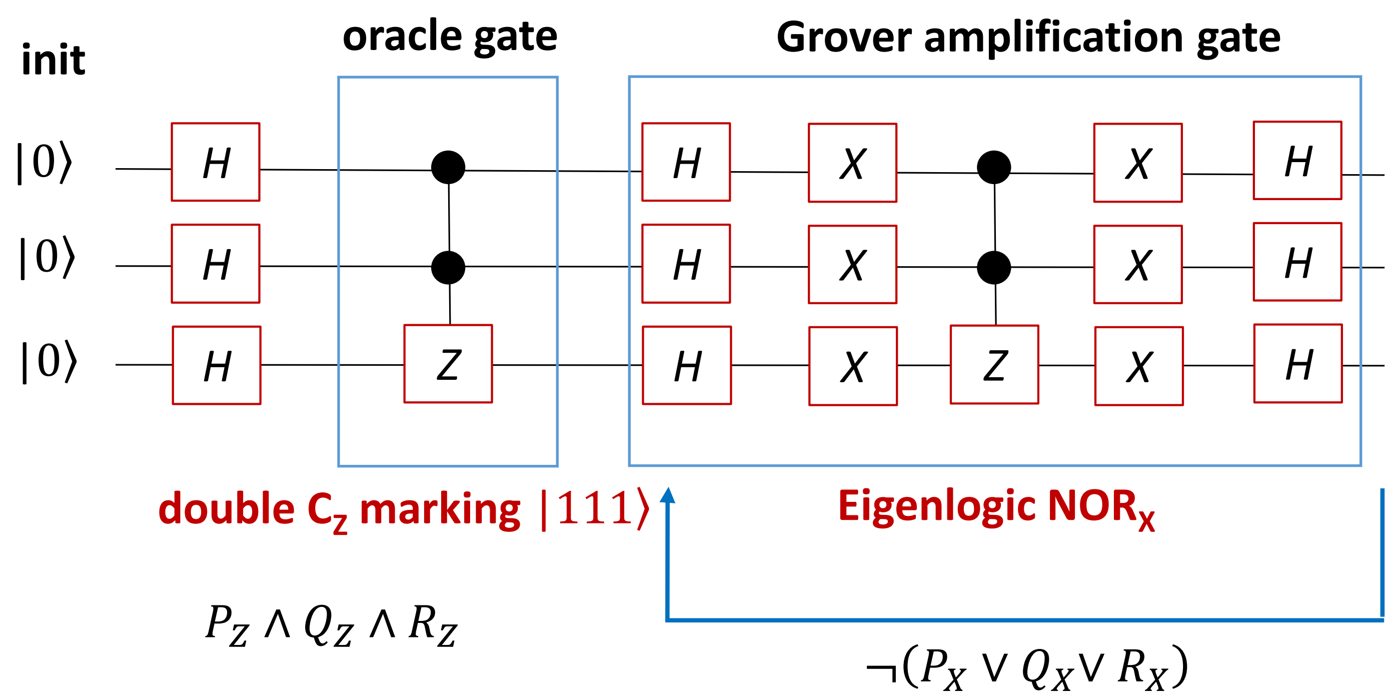
The Grover amplification gate corresponds to an Eigenlogic disjunction operator in the X system (in the circuit we have $\boldsymbol{G}_{\mathrm{NOR}} = -\boldsymbol{G}_{\mathrm{OR}}$). The oracle (phase) is here a **double control Z gate** (Eigenlogic 3-input AND $(-1)^{\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}\otimes\boldsymbol{\Pi}}$).

$\exists a\ P(a)$ decomposes, using **Skolemization** for <u>finite systems</u>

$\exists a \iff a_1 \vee a_2 \vee \cdots \vee a_N$

and

$\forall a \iff a_1 \wedge a_2 \wedge \cdots \wedge a_N$



Quantum Robot

Quantum robots: Paul Benioff

Paul Benioff was the first to propose the idea of a quantum Turing machine in 1980 **[1]**.

Benioff also gave the theoretical principle of a **quantum robot** in 1988 [2] as a first step towards a quantum mechanical description of systems that are aware of their environment and make decisions.

Currently a "Quantum Robotics" group has been created and a book has been published in 2017 [3].

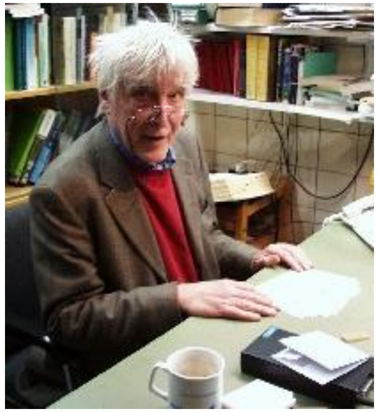
"Quantum Robotics is an emerging engineering and scientific research discipline that explores the application of quantum mechanics, quantum computing, quantum algorithms, and related fields to robotics. These developments are expected to impact the technical capability for robots to sense, plan, learn, and act in a dynamic environment."

[1] Benioff, P., 1980, **The computer as a physical system: A microscopic quantum mechanical Hamiltonian model of computers as represented by Turing machines**, J. of Statistical Phys., Vol. 22, (1980)

[2] Benioff, P., 1988, **Quantum Robots and Environments**, Physical Review A, Vol. 58, No.2, pp. 893–904

[3] Tandon, P., Lam, S., Shih, B., Mehta, T., Mitev, A., Ong, Z., **Quantum Robotics - A Primer on Current Science and Future Perspectives**, Morgan & Claypool. (2017)

Braitenberg Vehicles (BV)



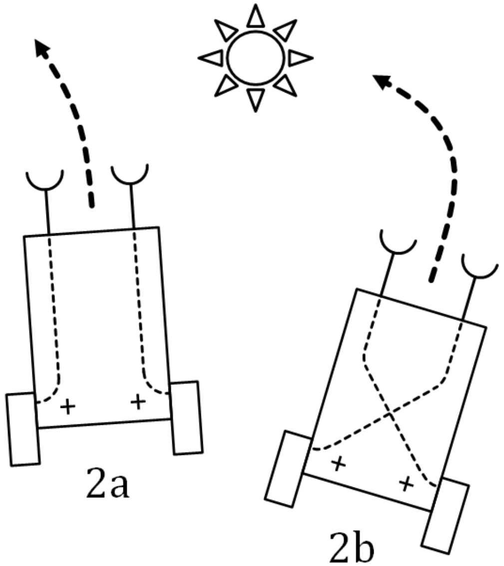
Valentino Braitenberg was a Cyberneticist. In his book *Vehicles* **[1]** various thought experiments using simple machines consisting in sensors, motors and wheels. Sensors detect light produced by surrounding sources.

The sensors can be connected in different configurations of combinations to the wheels. Simple changes in configuration can lead to complex and surprising results in the agent behavior.

We designed fuzzy logic *quantum-like* Braitenberg vehicles **[2]**.

The control is based on “Eigenlogic”.

The goal is to test the multiple combinations of logical gates used in the control of BV by analyzing their complex behavior.

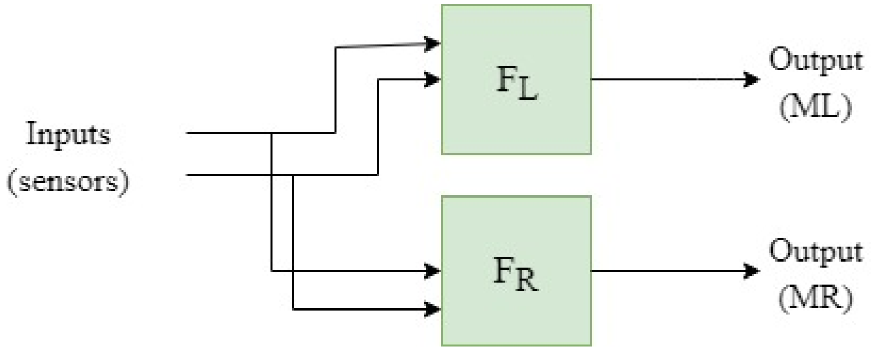


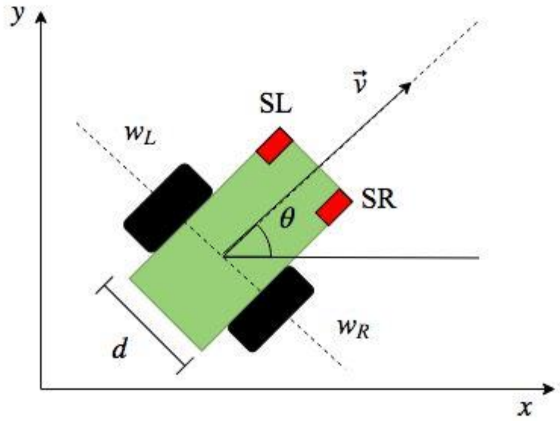
[1] Braitenberg, V. 1986, **Vehicles - Experiments in Synthetic Psychology**. MIT Press; Cambridge USA

[2] Z. Toffano, F. Dubois, **Quantum eigenlogic observables applied to the study of fuzzy behaviour of Braitenberg vehicle quantum robots**, Kybernetes, (2019)

Vehicle input-output structure [*]

The computational block is composed of logic operators (matrices) designed with the quantum-like Eigenlogic method. The control of the two-wheel motors (*ML*, *MR*), responds to signals from the two light sensors (*SL*, *SR*).





the input state vector representing the light intensity on *SL* and *SR* is: $|\psi\rangle = |\phi_{SL}\rangle \otimes |\phi_{SR}\rangle$

The fuzzy quantities for left and right wheel (*WL*, *WR*) control, are the mean values of the logical operators $\boldsymbol{F_L}$ and $\boldsymbol{F_R}$ on the input compound state $|\psi\rangle$:

$$\mu_L = \langle\psi|\boldsymbol{F_L}|\psi\rangle \quad , \quad \mu_R = \langle\psi|\boldsymbol{F_R}|\psi\rangle$$

[*] Cunha R.A.F., Sharma N., Toffano Z., Dubois F. **Fuzzy Logic Behavior of Quantum-Controlled Braitenberg Vehicle Agents**. In: Coecke B., Lambert-Mogiliansky A. (eds) Quantum Interaction. QI 2018. Lecture Notes in Computer Science, vol 11690. Springer, Cham. (2019)

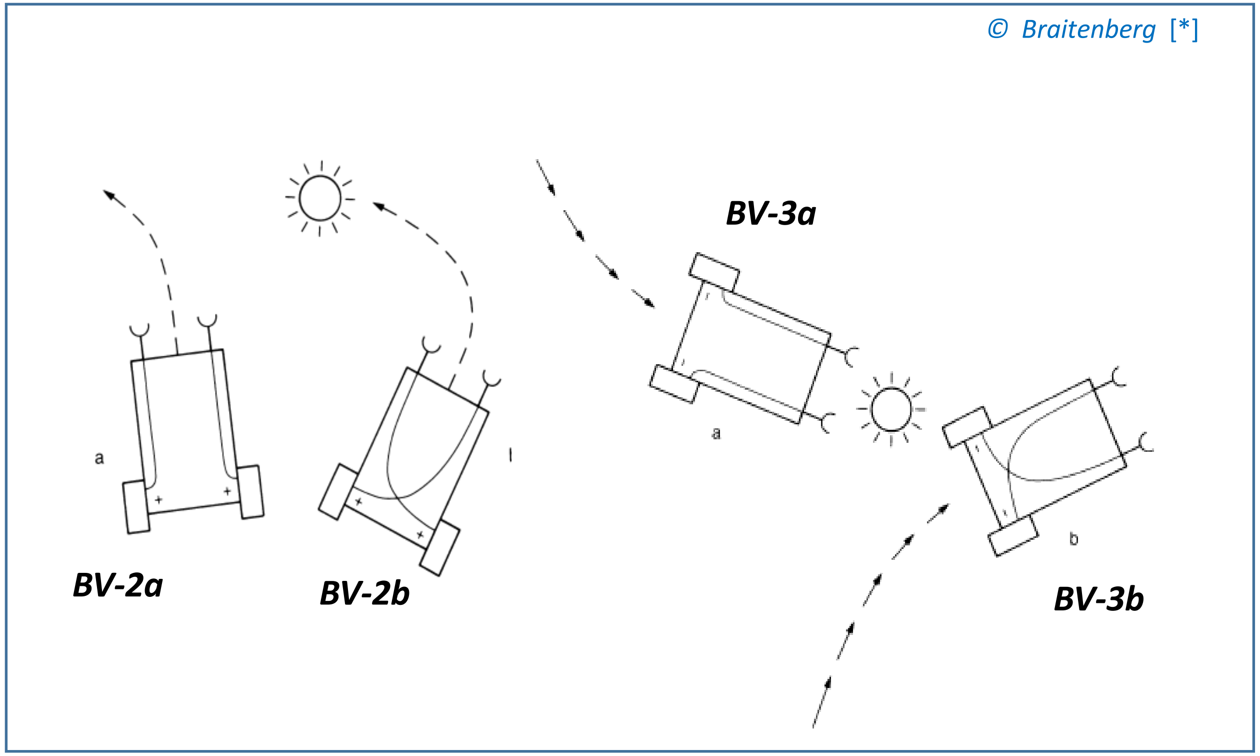
basic BV “classic” emotions using Eigenlogic

Fear *(BV-2a)* ($\boldsymbol{F}_L = \boldsymbol{A}$ and $\boldsymbol{F}_R = \boldsymbol{B}$): turns away from the light if one sensor is activated more than the other. If both are equal, the light source is “attacked”.

Aggression *(BV-2b)* ($\boldsymbol{F}_L = \boldsymbol{B}$ and $\boldsymbol{F}_R = \boldsymbol{A}$): when the light source is placed near either sensor, the vehicle will face it and go toward it.

Love *(BV-3a)* ($\boldsymbol{F}_L = \mathbb{I} - \boldsymbol{A}$ and $\boldsymbol{F}_R = \mathbb{I} - \boldsymbol{B}$): will go until it finds a light source, then slows to a stop. If one side sees light, the vehicle turns in the direction of the light.

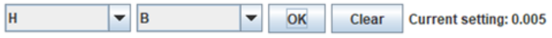
Exploration *(BV-3b)* ($\boldsymbol{F}_L = \mathbb{I} - \boldsymbol{B}$ and $\boldsymbol{F}_R = \mathbb{I} - \boldsymbol{A}$): will go to the nearby light source, keeps an eye open and will sail to other stronger sources, given the chance.

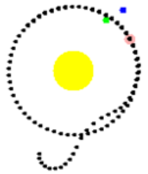


Quantum like emotions: worship , doubt

<u>Worship</u>: ($\boldsymbol{F}_L = \frac{1}{2}(1 - \boldsymbol{H} \otimes \boldsymbol{H})$ and $\boldsymbol{F}_R = \boldsymbol{B}$): one of the control logical operator is the projector version of the double-Hadamard qubit gate $\boldsymbol{H}$.

The vehicle keeps rotating around its own center in the absence of light. In the presence of light, it goes towards the source and starts to rotate around the source (or multiple sources when they are close together).

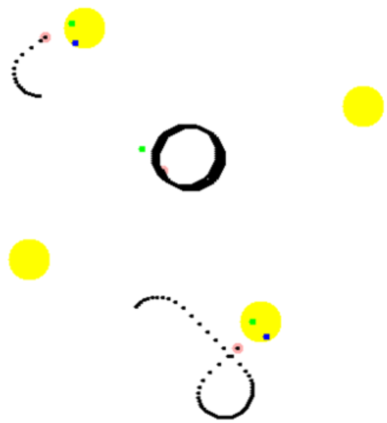




$\lvert x_f\rangle$	μ_L	μ_R	Behavior
$\lvert 00\rangle$	0.25	0	turns to the right slowly
$\lvert 01\rangle$	0.75	1	turns to the left slowly
$\lvert 10\rangle$	0.75	0	turns to the right
$\lvert 11\rangle$	0.25	1	turns to the left

<u>Doubt</u>: ($\boldsymbol{F}_L = \boldsymbol{B} \not\Rightarrow \boldsymbol{A}$ and $\boldsymbol{F}_R = \boldsymbol{F}_{XOR} = \frac{1}{2}(1 - \boldsymbol{Z} \otimes \boldsymbol{Z})$): here, one of the control logical operator is *XOR* (projector version of the double- $\boldsymbol{Z}$ qubit gate).

This operator provides a property that makes the vehicle turn around in circles, regardless of the presence or absence of stimuli.



$\lvert x_f\rangle$	μ_L	μ_R	Behavior
$\lvert 00\rangle$	0	0	no movement
$\lvert 01\rangle$	1	1	goes straight
$\lvert 10\rangle$	0	1	turns to the left
$\lvert 11\rangle$	0	0	no movement

quantum wheel of emotions

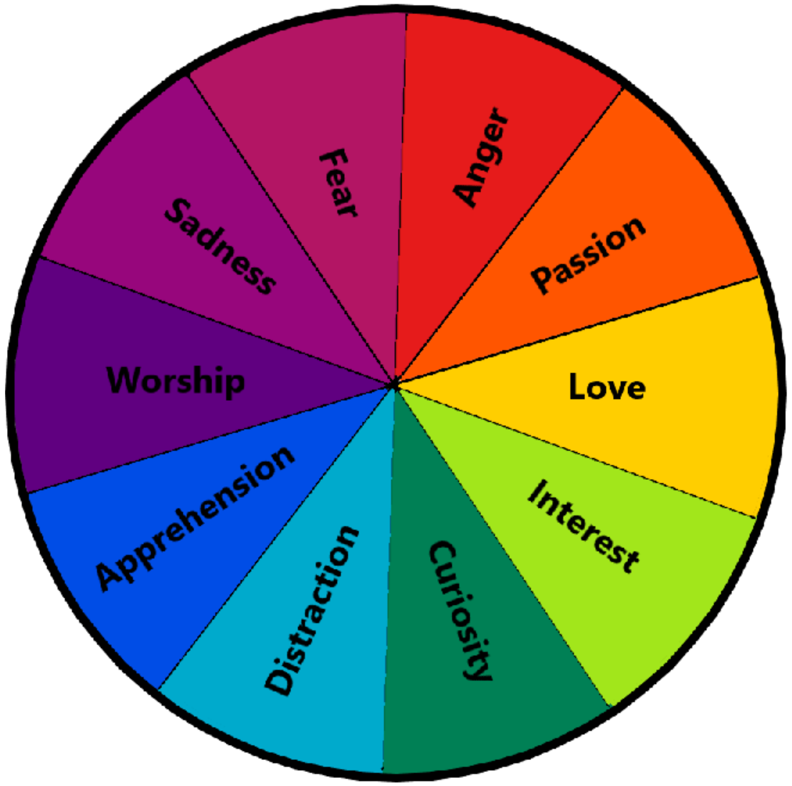
The concept of “wheel of emotions” introduced by Plutchik *et al.* **[*]** allows a continuous set of emotional states.

A small perturbation in the angle of the input state will correspond to small changes in the vehicle’s behavior.

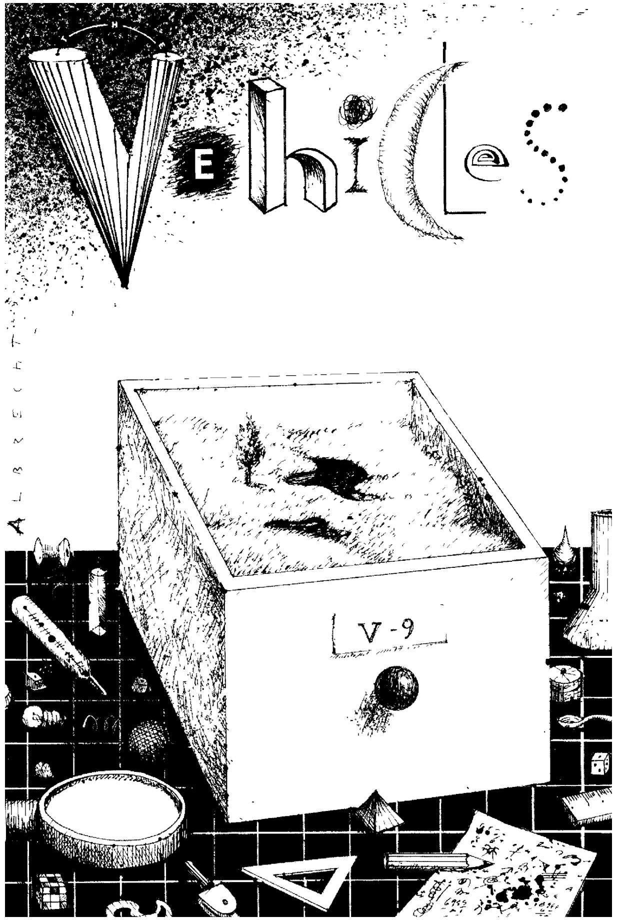
The emotions presented on the wheel shown are an example corresponding to the following $L \mid R$ logical control operators:

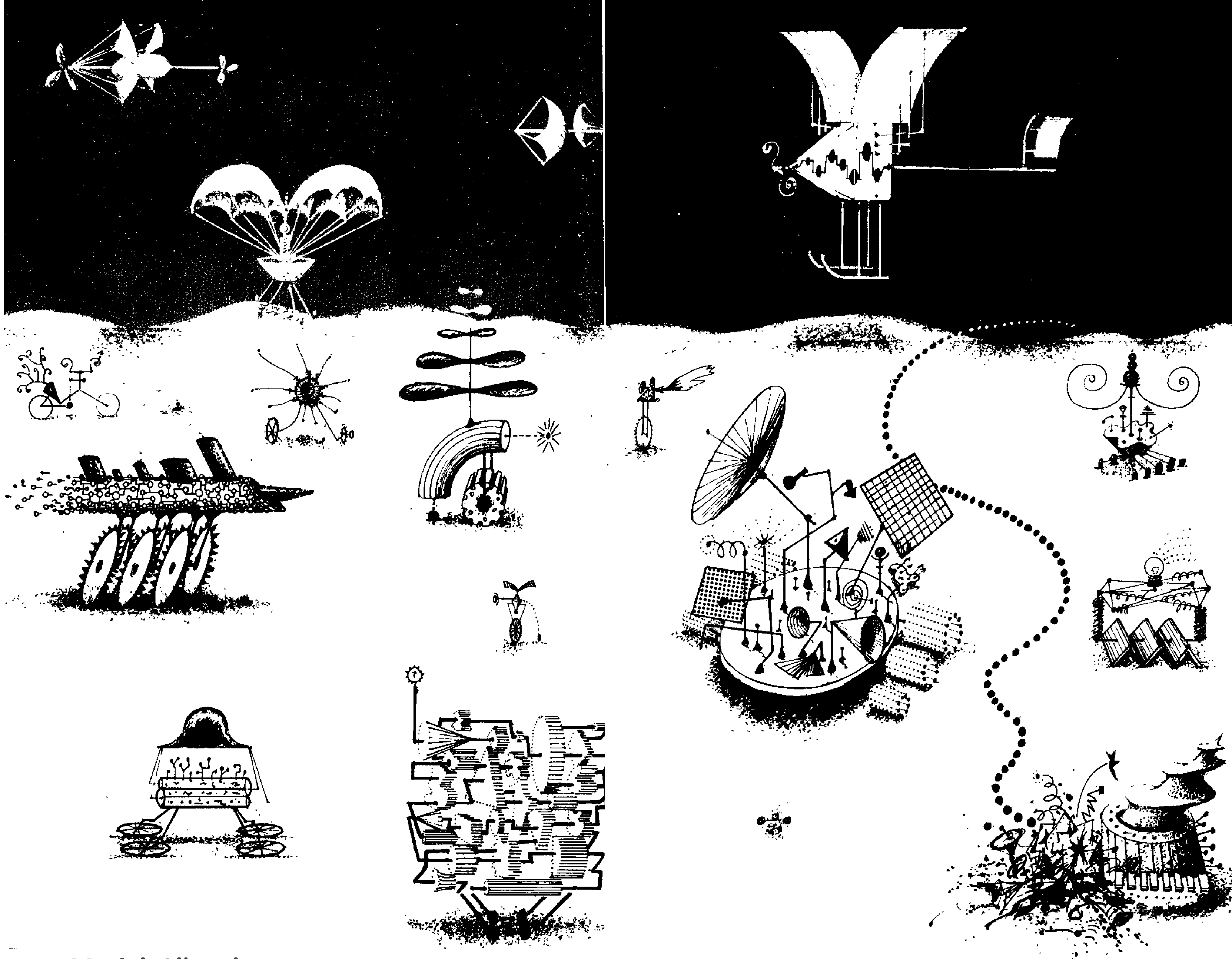
- (Anger-Aggressive) $B \mid A$
- (Passion) $B \nRightarrow A \mid A \nRightarrow B$
- (Love) $\bar{A} \mid \bar{B}$
- (Interest-Explore) $\bar{B} \mid \bar{A}$
- (Curiosity) $A \nRightarrow B \mid B \nRightarrow A$
- (Distraction) $A \nRightarrow B \mid XOR$
- (Apprehension) $B \nRightarrow A \mid XOR$
- (Worship) $\boldsymbol{H} \otimes \boldsymbol{H} \mid B$
- (Sadness) $CNOT \mid CNOT$
- (Fear) $A \mid B$

[*] Plutchik, R., 2001, **The Nature of Emotions**, Amercian Scientist, July-August, Vol. 89, N° 4, pp. 334-350



THANK -YOU





drawings by **Maciek Albrecht** © *Braitenberg, V. 1986, Vehicles: Experiments in Synthetic, Psychology. MIT Press*

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