# An introduction to quantum trajectories

Groupe Modélisation Quantique ISC-PIF
Paris, 20 mai 2019

Pierre Rouchon
Centre Automatique et Systèmes, Mines ParisTech, PSL Research University
Quantic Research Team (ENS, Inria, Mines)

# Outline


Quantum mechanics from scratch

Why density operator $\rho$ instead of wave function $|\psi\rangle$

Continuous-time formulations

Conclusion: two kinds of quantum feedback

Appendix: two key quantum systems
- Qubit (half-spin)
- Harmonic oscillator (spring)

Quantum mechanics from scratch

Why density operator $\rho$ instead of wave function $|\psi\rangle$

Continuous-time formulations

Conclusion: two kinds of quantum feedback

Appendix: two key quantum systems
- Qubit (half-spin)
- Harmonic oscillator (spring)

1. Schrödinger: wave funct. $|\psi\rangle \in \mathcal{H}$,

$$\frac{d}{dt}|\psi\rangle = -\tfrac{i}{\hbar}\boldsymbol{H}|\psi\rangle, \quad \boldsymbol{H} = \boldsymbol{H}_0 + u\boldsymbol{H}_1,$$

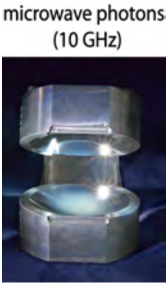
2. Origin of dissipation: collapse of the wave packet induced by the measurement of observable $\boldsymbol{O}$ with spectral decomp. $\sum_\mu \lambda_\mu \boldsymbol{P}_\mu$:
   - measurement outcome $\mu$ with proba. $\mathbb{P}_\mu = \langle\psi|\boldsymbol{P}_\mu|\psi\rangle$ depending on $|\psi\rangle$, just before the measurement
   - measurement back-action if outcome $\mu = y$:

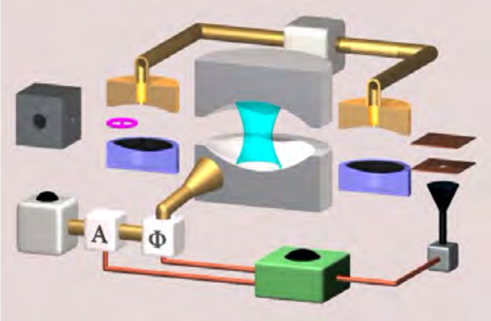
$$|\psi\rangle \mapsto |\psi\rangle_+ = \frac{\boldsymbol{P}_y|\psi\rangle}{\sqrt{\langle\psi|\boldsymbol{P}_y|\psi\rangle}}$$

3. Tensor product for the description of composite systems $(S, M)$:
   - Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$
   - Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_S \otimes \boldsymbol{I}_M + \boldsymbol{H}_{int} + \boldsymbol{I}_S \otimes \boldsymbol{H}_M$
   - observable on sub-system $M$ only: $\boldsymbol{O} = \boldsymbol{I}_S \otimes \boldsymbol{O}_M$.

[1] S. Haroche and J.M. Raimond. *Exploring the Quantum: Atoms, Cavities and Photons.* Oxford Graduate Texts, 2006.

**The first experimental realization of a quantum-state feedback:**




**Theory:** I. Dotsenko, . . . : Quantum feedback by discrete quantum non-demolition measurements: towards on-demand generation of photon-number states. Physical Review A, **2009**, 80: 013805-013813.
**Experiment:** C. Sayrin, . . . , **S. Haroche**:
Real-time quantum feedback prepares and stabilizes photon number states. Nature, **2011**, 477, 73-77.

- System $S$ corresponds to a quantized harmonic oscillator:

$$\mathcal{H}_S = \left\{ \sum_{n=0}^{\infty} \psi_n \; |n\rangle \;\middle|\; (\psi_n)_{n=0}^{\infty} \in l^2(\mathbb{C}) \right\},$$

where $|n\rangle$ is the photon-number state with $n$ photons ($\langle n_1 | n_2 \rangle = \delta_{n_1,n_2}$).

- Meter $M$ is a qubit, a 2-level system:

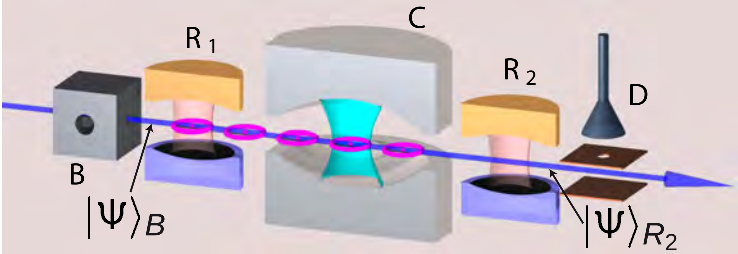
$$\mathcal{H}_M = \left\{ \psi_g \; |g\rangle + \psi_e \, |e\rangle \;\middle|\; \psi_g, \psi_e \in \mathbb{C} \right\},$$

where $|g\rangle$ (resp. $|e\rangle$) is the ground (resp. excited) state ($\langle g|g\rangle = \langle e|e\rangle = 1$ and $\langle g|e\rangle = 0$)

- **State of the composite system $|\Psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_M$:**

$$\begin{aligned} |\Psi\rangle &= \sum_{n\geq 0} \Big( \Psi_{ng} \; |n\rangle \otimes |g\rangle + \Psi_{ne} \; |n\rangle \otimes |e\rangle \Big) \\ &= \left( \sum_{n\geq 0} \Psi_{ng} \; |n\rangle \right) \otimes |g\rangle + \left( \sum_{n\geq 0} \Psi_{ne} \; |n\rangle \right) \otimes |e\rangle \,, \qquad \Psi_{ne}, \Psi_{ng} \in \mathbb{C}. \end{aligned}$$

Ortho-normal basis: $\big( |n\rangle \otimes |g\rangle \,, |n\rangle \otimes |e\rangle \big)_{n\in\mathbb{N}}$.




- When atom comes out $B$, the quantum state $|\boldsymbol{\Psi}\rangle_B$ of the composite system is separable: $|\boldsymbol{\Psi}\rangle_B = |\psi\rangle \otimes |g\rangle$.
- Just before the measurement in $D$, the state is in general entangled (not separable):

$$|\boldsymbol{\Psi}\rangle_{R_2} = \boldsymbol{U}_{SM}\Big( |\psi\rangle \otimes |g\rangle \Big) = \Big(\boldsymbol{M}_g |\psi\rangle \Big) \otimes |g\rangle + \Big(\boldsymbol{M}_e |\psi\rangle \Big) \otimes |e\rangle$$

where $\boldsymbol{U}_{SM} = \boldsymbol{U}_{R_2}\boldsymbol{U}_C\boldsymbol{U}_{R_1}$ is a unitary transformation (Schrödinger propagator) defining the measurement operators $\boldsymbol{M}_g$ and $\boldsymbol{M}_e$ on $\mathcal{H}_S$. Since $\boldsymbol{U}_{SM}$ is unitary, $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

Just before detector $D$ the quantum state is **entangled**:

$$|\mathbf{\Psi}\rangle_{R_2} = (\boldsymbol{M}_g |\psi\rangle) \otimes |g\rangle + (\boldsymbol{M}_e |\psi\rangle) \otimes |e\rangle$$

Just after outcome $y$, the state becomes **separable** [2]:

$$|\mathbf{\Psi}\rangle_D = \left( \frac{\boldsymbol{M}_y}{\sqrt{\left\langle \psi | \boldsymbol{M}_y^\dagger \boldsymbol{M}_y | \psi \right\rangle}} |\psi\rangle \right) \otimes |y\rangle .$$

Outcome $y$ obtained with probability $\mathbb{P}_y = \left\langle \psi | \boldsymbol{M}_y^\dagger \boldsymbol{M}_y | \psi \right\rangle$..

Quantum trajectories (Markov chain, stochastic dynamics):

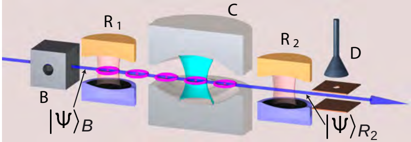
$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g}{\sqrt{\left\langle \psi_k | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_k \right\rangle}} |\psi_k\rangle , & y_k = g \text{ with probability } \left\langle \psi_k | \boldsymbol{M}_g^\dagger \boldsymbol{M}_g | \psi_k \right\rangle; \\ \frac{\boldsymbol{M}_e}{\sqrt{\left\langle \psi_k | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_k \right\rangle}} |\psi_k\rangle , & y_k = e \text{ with probability } \left\langle \psi_k | \boldsymbol{M}_e^\dagger \boldsymbol{M}_e | \psi_k \right\rangle; \end{cases}$$

with state $|\psi_k\rangle$ and measurement outcome $y_k \in \{g, e\}$ at time-step $k$:

[2] Measurement operator $\boldsymbol{O} = \boldsymbol{I}_S \otimes (|e\rangle \langle e| - |g\rangle \langle g|)$.

## Quantum Non Demolition (QND) measurement of photons [3]

$$|\boldsymbol{\Psi}\rangle_{R_2} = \boldsymbol{U}_{R_2}\boldsymbol{U}_C\boldsymbol{U}_{R_1}\big(\,|\psi\rangle\otimes|g\rangle\,\big)$$




$$\boldsymbol{U}_{R_1} = \boldsymbol{I}_S \otimes \left(\left(\frac{|g\rangle+|e\rangle}{\sqrt{2}}\right)\langle g| + \left(\frac{-|g\rangle+|e\rangle}{\sqrt{2}}\right)\langle e|\right)$$

$$\boldsymbol{U}_C = e^{-i\frac{\phi_0}{2}\boldsymbol{N}}\otimes|g\rangle\langle g| + e^{i\frac{\phi_0}{2}\boldsymbol{N}}\otimes|e\rangle\langle e|$$

$$\boldsymbol{U}_{R_2} = \boldsymbol{U}_{R_1}$$

$$\boldsymbol{U}_{R_1}\big(\,|\psi\rangle\otimes|g\rangle\,\big) = \tfrac{1}{\sqrt{2}}\left(|\psi\rangle\otimes|g\rangle + |\psi\rangle\otimes|e\rangle\right)$$

$$\boldsymbol{U}_C\boldsymbol{U}_{R_1}\big(\,|\psi\rangle\otimes|g\rangle\,\big) = \tfrac{1}{\sqrt{2}}\left(\left(e^{-i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle\right)\otimes|g\rangle + \left(e^{i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle\right)\otimes|e\rangle\right)$$

$$\begin{aligned}|\boldsymbol{\Psi}\rangle_{R_2} &= \tfrac{1}{2}\left(\left(e^{-i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle\right)\otimes(|g\rangle+|e\rangle) + \left(e^{i\frac{\phi_0}{2}\boldsymbol{N}}|\psi\rangle\right)\otimes(-|g\rangle+|e\rangle)\right)\\ &= \left(-i\sin(\tfrac{\phi_0}{2}\boldsymbol{N})\,|\psi\rangle\right)\otimes|g\rangle + \left(\cos(\tfrac{\phi_0}{2}\boldsymbol{N})\,|\psi\rangle\right)\otimes|e\rangle\end{aligned}$$

Thus $\boldsymbol{M}_g = -i\sin(\frac{\phi_0}{2}\boldsymbol{N})$ and $\boldsymbol{M}_e = \cos(\frac{\phi_0}{2}\boldsymbol{N})$.

Quantum Monte-Carlo simulations with MATLAB: `QNDphoton.m`

[3] M. Brune, . . . : Manipulation of photons in a cavity by dispersive atom-field coupling: quantum non-demolition measurements and generation of "Schrödinger cat" states . Physical Review A, 45:5193-5214, 1992.

# Outline

Quantum mechanics from scratch

Why density operator $\rho$ instead of wave function $|\psi\rangle$

Continuous-time formulations

Conclusion: two kinds of quantum feedback

Appendix: two key quantum systems
- Qubit (half-spin)
- Harmonic oscillator (spring)

# Why density operators $\boldsymbol{\rho}$ instead of wave functions $|\psi\rangle$

Consider once again the LKB photon-box:

$$|\psi_{k+1}\rangle = \begin{cases} \frac{\boldsymbol{M}_g}{\sqrt{\langle\psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\rangle}}|\psi_k\rangle, & y_k = g \text{ with probability } \langle\psi_k|\boldsymbol{M}_g^\dagger \boldsymbol{M}_g|\psi_k\rangle; \\ \frac{\boldsymbol{M}_e}{\sqrt{\langle\psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\rangle}}|\psi_k\rangle, & y_k = e \text{ with probability } \langle\psi_k|\boldsymbol{M}_e^\dagger \boldsymbol{M}_e|\psi_k\rangle; \end{cases}$$

Assume known $|\psi_0\rangle$ and **detector out of order** ($y = \varnothing$): **what about** $|\psi_1\rangle$ **?**

- Expectation value of $|\psi_1\rangle\langle\psi_1|$ knowing $|\psi_0\rangle$: [4]

$$\mathbb{E}\left(|\psi_1\rangle\langle\psi_1| \;\middle|\; |\psi_0\rangle\right) = \boldsymbol{M}_g|\psi_0\rangle\langle\psi_0|\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e|\psi_0\rangle\langle\psi_0|\boldsymbol{M}_e^\dagger.$$

- Set $\boldsymbol{K}(\boldsymbol{\rho}) \triangleq \boldsymbol{M}_g\boldsymbol{\rho}\boldsymbol{M}_g^\dagger + \boldsymbol{M}_e\boldsymbol{\rho}\boldsymbol{M}_e^\dagger$ for any operator $\boldsymbol{\rho}$.
- $\boldsymbol{\rho}_k$ expectation of $|\psi_k\rangle\langle\psi_k|$ knowing $|\psi_0\rangle$:

$$\boldsymbol{\rho}_{k+1} = \boldsymbol{K}(\boldsymbol{\rho}_k) \text{ and } \boldsymbol{\rho}_0 = |\psi_0\rangle\langle\psi_0|.$$

Linear map $\boldsymbol{K}$: trace preserving Kraus map (quantum channel).
Density operators $\boldsymbol{\rho}$: convex space of Hermitian non-negative operators of trace one.

[4] $|\psi\rangle\langle\psi|$: orthogonal projector on line spanned by unitary vector $|\psi\rangle$.

Detector efficiency $\boldsymbol{\eta} \in [0,1]$. Output $y \in \{g, e, \varnothing\}$:

$$\boldsymbol{\rho}_{k+1} = \begin{cases} \dfrac{\boldsymbol{K}_g(\boldsymbol{\rho}_k)}{\mathrm{Tr}\left(\boldsymbol{K}_g(\boldsymbol{\rho}_k)\right)}, \; y_k = g \text{ with probability } \; \mathrm{Tr}\left(\boldsymbol{K}_g(\boldsymbol{\rho}_k)\right); \\ \dfrac{\boldsymbol{K}_e(\boldsymbol{\rho}_k)}{\mathrm{Tr}\left(\boldsymbol{K}_e(\boldsymbol{\rho}_k)\right)}, \; y_k = e \text{ with probability } \; \mathrm{Tr}\left(\boldsymbol{K}_e(\boldsymbol{\rho}_k)\right); \\ \dfrac{\boldsymbol{K}_\varnothing(\boldsymbol{\rho}_k)}{\mathrm{Tr}\left(\boldsymbol{K}_\varnothing(\boldsymbol{\rho}_k)\right)}, \; y_k = \varnothing \text{ with probability } \; \mathrm{Tr}\left(\boldsymbol{K}_\varnothing(\boldsymbol{\rho}_k)\right); \end{cases}$$

with Kraus maps

$$\boldsymbol{K}_g(\boldsymbol{\rho}) = \boldsymbol{\eta} \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger, \qquad \boldsymbol{K}_e(\boldsymbol{\rho}) = \boldsymbol{\eta} \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger$$
$$\boldsymbol{K}_\varnothing(\boldsymbol{\rho}) = (1-\boldsymbol{\eta}) \left( \boldsymbol{M}_g \boldsymbol{\rho} \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \boldsymbol{\rho} \boldsymbol{M}_e^\dagger \right).$$

We still have:

$$\mathbb{E}\left(\boldsymbol{\rho}_{k+1} \mid \boldsymbol{\rho}_k\right) \triangleq \boldsymbol{K}(\boldsymbol{\rho}_k) = \boldsymbol{M}_g \boldsymbol{\rho}_k \boldsymbol{M}_g^\dagger + \boldsymbol{M}_e \boldsymbol{\rho}_k \boldsymbol{M}_e^\dagger = \sum_y \boldsymbol{K}_y(\boldsymbol{\rho}_k).$$

## Discrete-time quantum trajectories for open quantum systems

Four features:

1. Bayes law: $\mathbb{P}(\mu/y) = \mathbb{P}(y/\mu)\mathbb{P}(\mu) \;/\; \left(\sum_{\mu'} \mathbb{P}(y/\mu')\mathbb{P}(\mu')\right)$,
2. Schrödinger equations defining unitary transformations.
3. Partial collapse of the wave packet: irreversibility and dissipation are induced by the measurement of observables with degenerate spectra.
4. Tensor product for the description of composite systems.

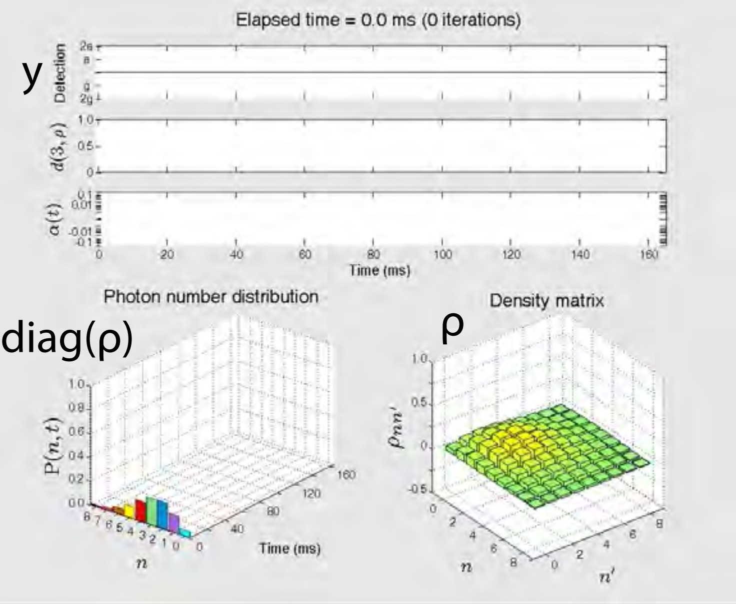
$\Rightarrow$ **Discrete-time Q. traj. :** Markov processes of state $\rho$, (density op.):
$\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{y,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\text{Tr}\left(\sum_{\mu=1}^{m} \eta_{y,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}$, with proba. $\mathbb{P}_y(\rho_k) = \sum_{\mu=1}^{m} \eta_{y,\mu} \text{ Tr}\left(\boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$
associated to Kraus maps [5] (ensemble average, quantum channel)

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

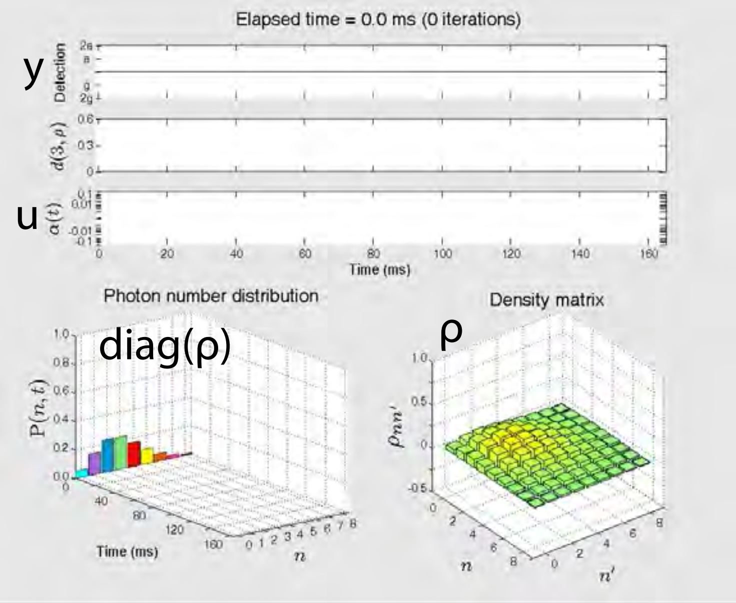
and left stochastic matrices (imperfections, decoherences) $(\eta_{y,\mu})$.

[5] M.A. Nielsen, I.L. Chuang: Quantum Computation and Quantum Information. Cambridge University Press, 2000.

## Open-loop dynamics ($u = 0$): experimental data



# Feedback stabilization around 3-photon state: experimental data



# Outline

Quantum mechanics from scratch

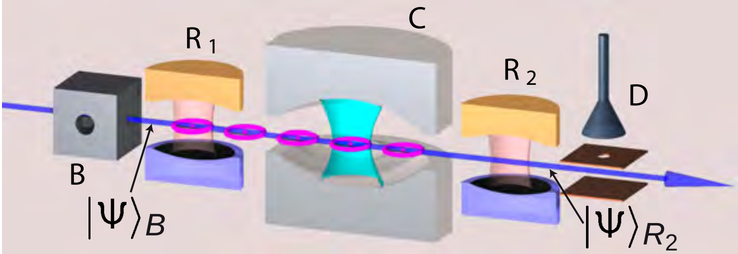
Why density operator $\rho$ instead of wave function $|\psi\rangle$

Continuous-time formulations

Conclusion: two kinds of quantum feedback

Appendix: two key quantum systems
- Qubit (half-spin)
- Harmonic oscillator (spring)

# Flashback to the LKB photon box




$$|\boldsymbol{\Psi}\rangle_{R_2} = \boldsymbol{U}_{SM}|\boldsymbol{\Psi}\rangle_B = \boldsymbol{U}_{SM}\big(\,|\psi\rangle \otimes |g\rangle\,\big) = \big(\boldsymbol{M}_g\,|\psi\rangle\,\big) \otimes |g\rangle + \big(\boldsymbol{M}_e\,|\psi\rangle\,\big) \otimes |e\rangle$$

with $\boldsymbol{M}_g^\dagger \boldsymbol{M}_g + \boldsymbol{M}_e^\dagger \boldsymbol{M}_e = \boldsymbol{I}$.

- Quantum trajectories (Markov chain, stochastic dynamics):

$$|\psi_{k+1}\rangle = \begin{cases} \dfrac{\boldsymbol{M}_g}{\sqrt{\langle\psi_k|\boldsymbol{M}_g^\dagger\boldsymbol{M}_g|\psi_k\rangle}}\,|\psi_k\rangle\,, & y_k = g \text{ with probability } \langle\psi_k|\boldsymbol{M}_g^\dagger\boldsymbol{M}_g|\psi_k\rangle; \\ \dfrac{\boldsymbol{M}_e}{\sqrt{\langle\psi_k|\boldsymbol{M}_e^\dagger\boldsymbol{M}_e|\psi_k\rangle}}\,|\psi_k\rangle\,, & y_k = e \text{ with probability } \langle\psi_k|\boldsymbol{M}_e^\dagger\boldsymbol{M}_e|\psi_k\rangle; \end{cases}$$

with state $|\psi_k\rangle$ and measurement outcome $y_k \in \{g, e\}$ at time-step $k$:

- The measurement outcome $y_k$ at discrete-time step $k$, is replaced by the small among of measurement signal $\boldsymbol{dy_t} \in \mathbb{R}$ obtained during an infinitesimal time interval $[t, t+dt]$.
- The measurement operator $\boldsymbol{M}_{y_k}$ becomes $\boldsymbol{M}_{\boldsymbol{dy_t}}$ close to identity:

$$\boldsymbol{M}_{\boldsymbol{dy_t}} = \boldsymbol{I} + \left(-\tfrac{i}{\hbar}\boldsymbol{H} - \tfrac{1}{2}\left(\boldsymbol{L}^\dagger \boldsymbol{L}\right)\right) dt + \boldsymbol{dy_t}\,\boldsymbol{L}$$

where operator $\boldsymbol{L}$ (not necessarily Hermitian) describes the measurement process and $\boldsymbol{H}$ is the Hamiltonian corresponding to the coherent evolution.
- The measurement backaction reads

$$|\psi\rangle_{t+dt} = \frac{\boldsymbol{M}_{\boldsymbol{dy_t}} |\psi\rangle_t}{\sqrt{\langle\psi|_t \boldsymbol{M}^\dagger_{\boldsymbol{dy_t}} \boldsymbol{M}_{\boldsymbol{dy_t}} |\psi\rangle_t}}$$

- Probability density of $\boldsymbol{dy} \in \mathbb{R}$ knowing $|\psi\rangle_t$: $\frac{e^{-\frac{\boldsymbol{dy}^2}{2dt}}}{\sqrt{2\pi dt}} \ \langle\psi|_t \boldsymbol{M}^\dagger_{\boldsymbol{dy}} \boldsymbol{M}_{\boldsymbol{dy}} |\psi\rangle_t$.

Coincides up to order $O(dt^{3/2})$ terms to $\boldsymbol{dy} = \langle\psi|_t (\boldsymbol{L} + \boldsymbol{L}^\dagger) |\psi\rangle_t \ dt + dW$ where $dW$ is a Wiener process (Gaussian of zero mean and variance $dt$).

Quantum Monte-Carlo simulations with MATLAB: `QNDqubit.m` ($\boldsymbol{L} = \boldsymbol{\sigma_z}$, $\boldsymbol{H} = 0$)

[6] For a mathematical exposure: A. Barchielli, M. Gregoratti: Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case. Springer Verlag,2009.

**Discrete-time models**: Markov chains

$$\rho_{k+1} = \frac{\sum_{\mu=1}^{m} \eta_{y,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger}{\mathrm{Tr}\left(\sum_{\mu=1}^{m} \eta_{y,\mu} \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)}, \text{ with proba. } \mathbb{P}_y(\rho_k) = \sum_{\mu=1}^{m} \eta_{y,\mu} \; \mathrm{Tr}\left(\boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger\right)$$

with ensemble averages corresponding to Kraus linear maps

$$\mathbb{E}\left(\rho_{k+1}|\rho_k\right) = \boldsymbol{K}(\rho_k) = \sum_\mu \boldsymbol{M}_\mu \rho_k \boldsymbol{M}_\mu^\dagger \quad \text{with} \quad \sum_\mu \boldsymbol{M}_\mu^\dagger \boldsymbol{M}_\mu = \boldsymbol{I}$$

**Continuous-time models**: stochastic differential systems [7]

$$d\rho_t = \left( -\tfrac{i}{\hbar}[\boldsymbol{H}, \rho_t] + \sum_\nu \boldsymbol{L}_\nu \rho_t \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)\right) dt$$
$$+ \sum_\nu \sqrt{\eta_\nu} \left( \boldsymbol{L}_\nu \rho_t + \rho_t \boldsymbol{L}_\nu^\dagger - \; \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\rho_t\right) \rho_t \right) dW_{\nu,t}$$

driven by Wiener processes $dW_{\nu,t}$, with measurements $y_{\nu,t}$,
$dy_{\nu,t} = \sqrt{\eta_\nu} \; \mathrm{Tr}\left((\boldsymbol{L}_\nu + \boldsymbol{L}_\nu^\dagger)\,\rho_t\right) \, dt + dW_{\nu,t}$, detection efficiencies $\eta_\nu \in [0,1]$ and Lindblad-Kossakowski master equations ($\eta_\nu \equiv 0$):

$$\frac{d}{dt}\rho = -\tfrac{i}{\hbar}[\boldsymbol{H}, \rho] + \sum_\nu \boldsymbol{L}_\nu \rho \boldsymbol{L}_\nu^\dagger - \frac{1}{2}(\boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu \rho + \rho \boldsymbol{L}_\nu^\dagger \boldsymbol{L}_\nu)$$

[7] A. Barchielli, M. Gregoratti: Quantum Trajectories and Measurements in Continuous Time: the Diffusive Case. Springer Verlag, 2009.

# Positivity-preserving formulation of diffusive SME [8]

With a single imperfect measurement $dy_t = \sqrt{\eta}\,\mathrm{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\,\rho_t\right)\,dt + dW_t$ and detection efficiency $\eta \in [0,1]$, the quantum state $\rho_t$ is usually mixed and obeys to

$$d\rho_t = \left(-\tfrac{i}{\hbar}[\boldsymbol{H},\rho_t] + \boldsymbol{L}\rho_t\boldsymbol{L}^\dagger - \frac{1}{2}(\boldsymbol{L}^\dagger\boldsymbol{L}\rho_t + \rho_t\boldsymbol{L}^\dagger\boldsymbol{L})\right)dt$$
$$+\sqrt{\eta}\left(\boldsymbol{L}\rho_t + \rho_t\boldsymbol{L}^\dagger - \mathrm{Tr}\left((\boldsymbol{L}+\boldsymbol{L}^\dagger)\rho_t\right)\rho_t\right)dW_t$$

driven by the Wiener process $dW_t$

With Itō rules, it can be written as the following "discrete-time" Markov model

$$\rho_{t+dt} = \frac{\boldsymbol{M}_{dy_t}\rho_t\boldsymbol{M}_{dy_t}^\dagger + (1-\eta)\boldsymbol{L}\rho_t\boldsymbol{L}^\dagger dt}{\mathrm{Tr}\left(\boldsymbol{M}_{dy_t}\rho_t\boldsymbol{M}_{dy_t}^\dagger + (1-\eta)\boldsymbol{L}\rho_t\boldsymbol{L}^\dagger dt\right)}$$

with $\boldsymbol{M}_{dy_t} = \boldsymbol{I} + \left(-\frac{i}{\hbar}\boldsymbol{H} - \frac{1}{2}\left(\boldsymbol{L}^\dagger\boldsymbol{L}\right)\right)dt + \sqrt{\eta}\,dy_t\boldsymbol{L}$.

**$\rho_0$ density operator $\mapsto$ for all $t>0$, $\rho_t$ density operator**

[8] Such SME precisely describe cutting-edge experiments with superconducting qubits under homodyne and heterodyne continuous-time measurements. See, e.g., the group of Benjamin Huard at ENS-Lyon: `http://www.physinfo.fr/index.html`.

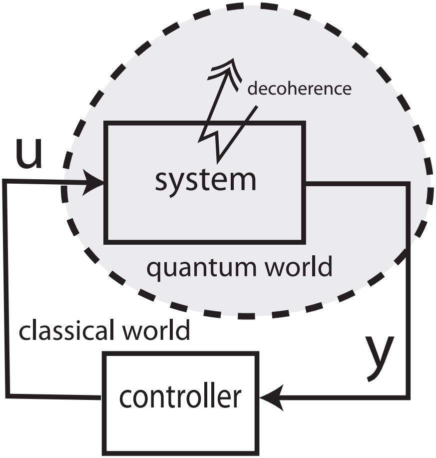
Quantum mechanics from scratch

Why density operator $\rho$ instead of wave function $|\psi\rangle$

Continuous-time formulations

Conclusion: two kinds of quantum feedback

Appendix: two key quantum systems
- Qubit (half-spin)
- Harmonic oscillator (spring)




P-controller (Markovian feedback [a]):
for $u_t \, dt = k \, dy_t$, average closed-loop dynamics of $\rho$ remains governed by a Lindblad master equation.

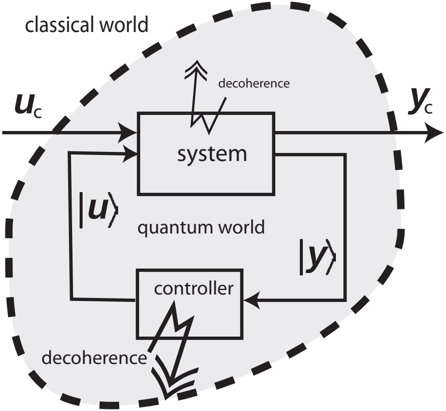
PID controller: no Lindblad master equation in closed-loop;

Nonlinear hidden-state stochastic systems: convergence analysis, Lyapunov exponents, dynamic output feedback, delays, robustness, . . .

[a] H.M. Wiseman: Quantum Trajectories and Feedback. PhD Thesis, University of Queensland, 1994.

**Short sampling times limit feedback complexity**

Quantum analogue of Watt speed governor: a **dissipative** mechanical system controls another mechanical system [9]



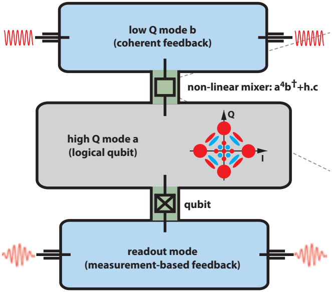

Optical pumping (Kastler 1950), coherent population trapping (Arimondo 1996)

Dissipation engineering, autonomous feedback: (Zoller, Cirac, Wolf, Verstraete, Devoret, Schoelkopf, Siddiqi, Lloyd, Viola, Ticozzi, Leghtas, Mirrahimi, Sarlette, ...)

**(S,L,H) theory** and linear quantum systems: quantum feedback networks based on stochastic Schrödinger equation, Heisenberg picture (Gardiner, Yurke, Mabuchi, Genoni, Serafini, Milburn, Wiseman, Doherty, Gough, James, Petersen, Nurdin, Yamamoto, Zhang, Dong, . . . )

Stability analysis: Kraus maps and Lindblad propagators are always contractions (non commutative diffusion and consensus).

[9] J.C. Maxwell: On governors. Proc. of the Royal Society, No.100, 1868.



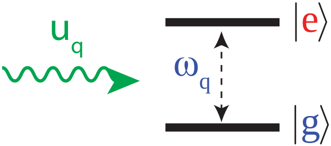

• Quantic in Paris[a]: 3 theoreticians, 1 experimentalist, 4 PhD, 2 PostDocs.
• Development of theoretical methods and experimental devices ensuring robust processing of quantum information.

[a] https://team.inria.fr/quantic/

• Address Quantum Error Correction (QEC) in a new direction[10]: instead of relying on a large number of physical qubits and collective syndrome measurements to obtain a logical qubit, engineer a logical qubit of tunable high fidelity, localized in a single harmonic oscillator (**cat qubit**), relying on measurement-based and coherent feedback schemes, exploiting typical nonlinearities of Josephson superconducting circuits, and subject essentially to one error channel (finite photon life-time).

[10] M. Mirrahimi, Z. Leghtas, V.V. Albert, S. Touzard, R.J. Schoelkopf, L. Jiang, and M.H. Devoret. Dynamically protected cat-qubits: a new paradigm for universal quantum computation. New Journal of Physics, 16:045014, 2014.

- Hilbert space:
  $\mathcal{H}_M = \mathbb{C}^2 = \left\{ c_g \left| g \right\rangle + c_e \left| e \right\rangle,\ c_g, c_e \in \mathbb{C} \right\}.$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_M), \rho^\dagger = \rho,\ \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$ .
- Operators and commutations:
  $\boldsymbol{\sigma_-} = \left| g \right\rangle \left\langle e \right|,\ \boldsymbol{\sigma_+} = \boldsymbol{\sigma_-}^\dagger = \left| e \right\rangle \left\langle g \right|$
  $\boldsymbol{\sigma_x} = \boldsymbol{\sigma_-} + \boldsymbol{\sigma_+} = \left| g \right\rangle \left\langle e \right| + \left| e \right\rangle \left\langle g \right|;$
  $\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_-} - i\boldsymbol{\sigma_+} = i \left| g \right\rangle \left\langle e \right| - i \left| e \right\rangle \left\langle g \right|;$
  $\boldsymbol{\sigma_z} = \boldsymbol{\sigma_+}\boldsymbol{\sigma_-} - \boldsymbol{\sigma_-}\boldsymbol{\sigma_+} = \left| e \right\rangle\left\langle e \right| - \left| g \right\rangle\left\langle g \right|;$
  $\boldsymbol{\sigma_x}^2 = \boldsymbol{I},\ \boldsymbol{\sigma_x}\boldsymbol{\sigma_y} = i\boldsymbol{\sigma_z},\ [\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}] = 2i\boldsymbol{\sigma_z},\ \ldots$




- Hamiltonian: $\boldsymbol{H}_M/\hbar = \omega_q \boldsymbol{\sigma_z}/2 + \boldsymbol{u_q}\boldsymbol{\sigma_x}.$
- Bloch sphere representation:
  $\mathcal{D} = \left\{ \frac{1}{2}\left(\boldsymbol{I} + x\boldsymbol{\sigma_x} + y\boldsymbol{\sigma_y} + z\boldsymbol{\sigma_z}\right) \,\middle|\, (x,y,z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 \leq 1 \right\}$

[11] See S. M. Barnett, P.M. Radmore: Methods in Theoretical Quantum Optics. Oxford University Press, 2003.

# Quantum harmonic oscillator (spring system) [11]

- Hilbert space:
  $\mathcal{H}_S = \left\{ \sum_{n\geq 0} \psi_n \left|n\right\rangle, \; (\psi_n)_{n\geq 0} \in l^2(\mathbb{C}) \right\} \equiv L^2(\mathbb{R}, \mathbb{C})$
- Quantum state space:
  $\mathcal{D} = \{\rho \in \mathcal{L}(\mathcal{H}_S), \rho^\dagger = \rho, \; \mathrm{Tr}\,(\rho) = 1, \rho \geq 0\}$ .
- Operators and commutations:
  $\boldsymbol{a} \left|n\right\rangle = \sqrt{n} \left|\boldsymbol{n\text{-}1}\right\rangle$, $\boldsymbol{a}^\dagger \left|n\right\rangle = \sqrt{n+1} \left|n+1\right\rangle$;
  $\boldsymbol{N} = \boldsymbol{a}^\dagger \boldsymbol{a}$, $\boldsymbol{N} \left|n\right\rangle = n \left|n\right\rangle$;
  $[\boldsymbol{a}, \boldsymbol{a}^\dagger] = \boldsymbol{I}$, $\boldsymbol{a} f(\boldsymbol{N}) = f(\boldsymbol{N} + \boldsymbol{I})\boldsymbol{a}$;
  $\boldsymbol{D}_\alpha = e^{\alpha \boldsymbol{a}^\dagger - \alpha^\dagger \boldsymbol{a}}$.
  $\boldsymbol{a} = \boldsymbol{X} + i\boldsymbol{P} = \frac{1}{\sqrt{2}}\left(x + \frac{\partial}{\partial x}\right)$, $[\boldsymbol{X}, \boldsymbol{P}] = \imath \boldsymbol{I}/2$.
- Hamiltonian: $\boldsymbol{H}_S/\hbar = \omega_c \boldsymbol{a}^\dagger \boldsymbol{a} + \boldsymbol{u_c}(\boldsymbol{a} + \boldsymbol{a}^\dagger)$.
  (associated classical dynamics:
  $\frac{dx}{dt} = \omega_c p$, $\frac{dp}{dt} = -\omega_c x - \sqrt{2} u_c$).
- Classical pure state $\equiv$ coherent state $\left|\alpha\right\rangle$
  $\alpha \in \mathbb{C}$ : $\left|\alpha\right\rangle = \sum_{n\geq 0} \left( e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \right) \left|n\right\rangle$; $\left|\alpha\right\rangle \equiv \frac{1}{\pi^{1/4}} e^{\imath \sqrt{2} x \Im \alpha} e^{-\frac{(x - \sqrt{2}\Re\alpha)^2}{2}}$
  $\boldsymbol{a} \left|\alpha\right\rangle = \alpha \left|\alpha\right\rangle$, $\boldsymbol{D}_\alpha \left|0\right\rangle = \left|\alpha\right\rangle$.