A QUANTUM FORMALISM OF DYNAMICAL SYSTEMS

Symposium 11: Mathematical modeling of complex systems

Author: Joan C. Micó.

1. The Hamiltonian Dirac approach of dynamical systems.
2. The Schrödinger equation.
3. The probabilistic interpretation.
4. The one-dimensional time-independent Schrödinger equation.
5. Application case: the logistic function.

1. The Hamiltonian Dirac approach of dynamical systems.

Hamiltonian:

$$H_T(t,\boldsymbol{q},\boldsymbol{p}) = V(t,\boldsymbol{q}) + f_m(t,\boldsymbol{q})(p_m - g_m(t,\boldsymbol{q}))$$

Energy E :

$$E = V(\boldsymbol{q}) + f_m(\boldsymbol{q})(p_m - g_m(\boldsymbol{q}))$$

Hamilton-Jacobi equation for the Action $S(t,\boldsymbol{q})$:

$$\frac{\partial S(t,\boldsymbol{q})}{\partial t} + f_m(t,\boldsymbol{q})\left(\frac{\partial S(t,\boldsymbol{q})}{\partial q_m} - g_m(t,\boldsymbol{q})\right) + V(t,\boldsymbol{q}) = 0$$

2. The Schrödinger equation.

Schrödinger equation:
General approach

$$i\sigma\frac{\partial\Psi(t,\boldsymbol{q})}{\partial t} = \widehat{H}(t,\widehat{\boldsymbol{q}},\widehat{\boldsymbol{p}})\,\Psi(t,\boldsymbol{q})$$

(σ is the system Planck constant)

For the particular Hamiltonian of dynamical systems H_T :

$$\widehat{H}(t,\widehat{\boldsymbol{q}},\widehat{\boldsymbol{p}})\,\Psi(t,\boldsymbol{q}) = \frac{1}{2}\left(f_i(t,\widehat{\boldsymbol{q}})\hat{p}_i + \hat{p}_i f_i(t,\widehat{\boldsymbol{q}})_i\right)\Psi(t,\boldsymbol{q}) + W(t,\widehat{\boldsymbol{q}})\,\Psi(t,\boldsymbol{q})$$

With: $W(t,\widehat{\boldsymbol{q}}) = V(t,\widehat{\boldsymbol{q}}) - f_i(t,\widehat{\boldsymbol{q}})\,g_i(t,\widehat{\boldsymbol{q}})$

And the quantization rules: $ff(t,\widehat{\boldsymbol{q}})\,\Psi(t,\boldsymbol{q}) = ff(t,\boldsymbol{q})\,\Psi(t,\boldsymbol{q})$ $\quad \hat{p}_i\,\Psi(t,\boldsymbol{q}) = -i\sigma\frac{\partial\Psi(t,\boldsymbol{q})}{\partial q_i}$

(**ff** arbitrary)

(after some computations)

Schrödinger equation (of dynamical systems):

$$i\sigma\frac{\partial\Psi(t,\boldsymbol{q})}{\partial t} = -i\sigma\boldsymbol{f}(t,\boldsymbol{q})\boldsymbol{\nabla}\Psi(t,\boldsymbol{q}) - i\frac{\sigma}{2}\big(\boldsymbol{\nabla}\boldsymbol{f}(t,\boldsymbol{q})\big)\Psi(t,\boldsymbol{q}) + W(t,\boldsymbol{q})\,\Psi(t,\boldsymbol{q})$$

time-independent Hamiltonian $\longrightarrow$ $\longleftarrow$ $\Psi(t,\boldsymbol{q}) = e^{-i\frac{E}{\sigma}t}\,\psi(\boldsymbol{q})$

Time-independent Schrödinger equation :

$$\boldsymbol{f}(\boldsymbol{q})\boldsymbol{\nabla}\psi(\boldsymbol{q}) = \left(\frac{i}{\sigma}\big(E - W(\boldsymbol{q})\big) - \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{f}(\boldsymbol{q})\right)\psi(\boldsymbol{q})$$

3. The probabilistic interpretation.

Schrödinger equation:

$$i\sigma\frac{\partial\Psi(t,\boldsymbol{q})}{\partial t} = -i\sigma\boldsymbol{f}(t,\boldsymbol{q})\boldsymbol{\nabla}\Psi(t,\boldsymbol{q}) - i\frac{\sigma}{2}\big(\boldsymbol{\nabla}\boldsymbol{f}(t,\boldsymbol{q})\big)\Psi(t,\boldsymbol{q}) + W(t,\boldsymbol{q})\,\Psi(t,\boldsymbol{q})$$

$$\Psi(t,\boldsymbol{q}) = \mathrm{A}(t,\boldsymbol{q})\,e^{i\frac{\mathrm{B}(t,\boldsymbol{q})}{\sigma}}$$

$$\frac{\partial\mathrm{B}(t,\boldsymbol{q})}{\partial t} + \boldsymbol{f}(t,\boldsymbol{q})\boldsymbol{\nabla}\mathrm{B}(t,\boldsymbol{q}) + \mathrm{W}(t,\boldsymbol{q}) = 0$$

The **phase** $\mathrm{B}(t,\boldsymbol{q})$ holds the **Hamilton-Jacobi equation**

$$\frac{\partial\mathrm{A}^2(t,\boldsymbol{q})}{\partial t} + \boldsymbol{\nabla}\big(\boldsymbol{f}(t,\boldsymbol{q})\mathrm{A}^2(t,\boldsymbol{q})\big) = 0$$

The **square amplitude** $\mathrm{A}^2(t,\boldsymbol{q}) = |\Psi(t,\boldsymbol{q})|^2$ holds the **probability conservation law** with $\boldsymbol{f}(t,\boldsymbol{q})$ the corresponding **probability current density**

4. The one-dimensional time-independent Schrödinger equation.

$$f(q)\psi'(q) = \left(\frac{i}{\sigma}\big(E - W(q)\big) - \frac{1}{2}f'(q)\right)\psi(q) \qquad W(q) = V(q) - f(q)g(q)$$

$$\Downarrow$$

Solution: $$\psi(q) = \frac{\psi_0}{\sqrt{f(q)}}\mathrm{e}^{\frac{i}{\sigma}\left(\varphi + E\int\frac{dq}{f(q)} - \int\frac{W(q)}{f(q)}dq\right)}$$

With the Dirac consistency equation:

$$f(q)\,V''(q) = 0 \longrightarrow W(q) = k\,q - f(q)\,g(q)$$

$$\Downarrow$$

Solution undetermined by the $g(q)$ function

$$\psi(q) = \frac{\psi_0}{\sqrt{f(q)}}\mathrm{e}^{\frac{i}{\sigma}\left(\varphi + E\int\frac{dq}{f(q)} - k\int\frac{q}{f(q)}dq + \int g(q)dq\right)}$$

5. Application case: the logistic function.

$$\psi(q) = \frac{\psi_0}{\sqrt{f(q)}} e^{\frac{i}{\sigma}\left(\varphi + E\int\frac{dq}{f(q)} - k\int\frac{q}{f(q)}dq + \int g(q)dq\right)}$$

$$f(q) = a\,q - b\,q^2 = q(a - b\,q)$$

$$\psi(q) = \begin{cases} \dfrac{\psi_0}{\sqrt{-q(a-b\,q)}} e^{-\frac{\pi E}{\sigma a} + \frac{i}{\sigma}\left(\frac{\pi}{2} + \varphi + \frac{E}{a} ln(-q) + \left(\frac{k}{b} - \frac{E}{a}\right) ln(a - b\,q) + \int g(q)dq\right)} : q < 0 \\ \dfrac{\psi_0}{\sqrt{q(a-b\,q)}} e^{\frac{i}{\sigma}\left(\varphi + \frac{E}{a} ln q + \left(\frac{k}{b} - \frac{E}{a}\right) ln(a - b\,q) + \int g(q)dq\right)} : 0 < q < \dfrac{a}{b} \\ \dfrac{\psi_0}{\sqrt{q(b\,q-a)}} e^{-\frac{\pi}{\sigma}\left(\frac{k}{b} - \frac{E}{a}\right) + \frac{i}{\sigma}\left(\frac{\pi}{2} + \varphi + \frac{E}{a} ln q + \left(\frac{k}{b} - \frac{E}{a}\right) ln(b\,q - a) + \int g(q)dq\right)} : q > \dfrac{a}{b} \end{cases}$$

The research is centered in finding the mathematical pattern of the $g(q)$ function that permits to undo the singularities and that provides the quantization of the system Energy E.