

# A QUANTUM FORMALISM OF DYNAMICAL SYSTEMS

**Symposium 11: Mathematical modeling of complex systems**

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1. The Hamiltonian Dirac approach of dynamical systems.
2. The Schrödinger equation.
3. The probabilistic interpretation.
4. The one-dimensional time-independent Schrödinger equation.
5. Application case: the logistic function.

# 1. The Hamiltonian Dirac approach of dynamical systems.

**Hamiltonian:**

$$H_T(t, \mathbf{q}, \mathbf{p}) = V(t, \mathbf{q}) + f_m(t, \mathbf{q})(p_m - g_m(t, \mathbf{q}))$$

**Energy E :**

$$E = V(\mathbf{q}) + f_m(\mathbf{q})(p_m - g_m(\mathbf{q}))$$

**Hamilton-Jacobi equation** for the Action  $S(t, \mathbf{q})$ :

$$\frac{\partial S(t, \mathbf{q})}{\partial t} + f_m(t, \mathbf{q}) \left( \frac{\partial S(t, \mathbf{q})}{\partial q_m} - g_m(t, \mathbf{q}) \right) + V(t, \mathbf{q}) = 0$$

## 2. The Schrödinger equation.

Schrödinger equation: 
$$i\sigma \frac{\partial \Psi(t, \mathbf{q})}{\partial t} = \hat{H}(t, \hat{\mathbf{q}}, \hat{\mathbf{p}}) \Psi(t, \mathbf{q})$$
 ( $\sigma$  is the system Planck constant)

General approach

For the particular Hamiltonian of dynamical systems  $H_T$  :

$$\hat{H}(t, \hat{\mathbf{q}}, \hat{\mathbf{p}}) \Psi(t, \mathbf{q}) = \frac{1}{2} (f_i(t, \hat{\mathbf{q}}) \hat{p}_i + \hat{p}_i f_i(t, \hat{\mathbf{q}})_i) \Psi(t, \mathbf{q}) + W(t, \hat{\mathbf{q}}) \Psi(t, \mathbf{q})$$

With:  $W(t, \hat{\mathbf{q}}) = V(t, \hat{\mathbf{q}}) - f_i(t, \hat{\mathbf{q}}) g_i(t, \hat{\mathbf{q}})$

And the quantization rules:  $f f(t, \hat{\mathbf{q}}) \Psi(t, \mathbf{q}) = f f(t, \mathbf{q}) \Psi(t, \mathbf{q})$   $\hat{p}_i \Psi(t, \mathbf{q}) = -i\sigma \frac{\partial \Psi(t, \mathbf{q})}{\partial q_i}$

( $f$  arbitrary)

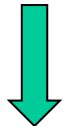


(after some computations)

**Schrödinger equation** (of dynamical systems):

$$i\sigma \frac{\partial \Psi(t, \mathbf{q})}{\partial t} = -i\sigma f(t, \mathbf{q}) \nabla \Psi(t, \mathbf{q}) - i \frac{\sigma}{2} (\nabla f(t, \mathbf{q})) \Psi(t, \mathbf{q}) + W(t, \mathbf{q}) \Psi(t, \mathbf{q})$$

time-independent Hamiltonian  $\longrightarrow$   $\Psi(t, \mathbf{q}) = e^{-i \frac{E}{\sigma} t} \psi(\mathbf{q})$



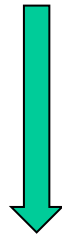
**Time-independent Schrödinger equation :**

$$f(\mathbf{q}) \nabla \psi(\mathbf{q}) = \left( \frac{i}{\sigma} (E - W(\mathbf{q})) - \frac{1}{2} \nabla f(\mathbf{q}) \right) \psi(\mathbf{q})$$

### 3. The probabilistic interpretation.

**Schrödinger equation:**

$$i\sigma \frac{\partial \Psi(t, \mathbf{q})}{\partial t} = -i\sigma \mathbf{f}(t, \mathbf{q}) \nabla \Psi(t, \mathbf{q}) - i \frac{\sigma}{2} (\nabla \mathbf{f}(t, \mathbf{q})) \Psi(t, \mathbf{q}) + W(t, \mathbf{q}) \Psi(t, \mathbf{q})$$



$$\Psi(t, \mathbf{q}) = A(t, \mathbf{q}) e^{i \frac{B(t, \mathbf{q})}{\sigma}}$$

$$\frac{\partial B(t, \mathbf{q})}{\partial t} + \mathbf{f}(t, \mathbf{q}) \nabla B(t, \mathbf{q}) + W(t, \mathbf{q}) = 0$$

← The **phase**  $B(t, \mathbf{q})$  holds the **Hamilton-Jacobi equation**

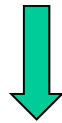
$$\frac{\partial A^2(t, \mathbf{q})}{\partial t} + \nabla (\mathbf{f}(t, \mathbf{q}) A^2(t, \mathbf{q})) = 0$$



The **square amplitude**  $A^2(t, \mathbf{q}) = |\Psi(t, \mathbf{q})|^2$  holds the **probability conservation law** with  $\mathbf{f}(t, \mathbf{q})$  the corresponding **probability current density**

## 4. The one-dimensional time-independent Schrödinger equation.

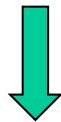
$$f(q)\psi'(q) = \left( \frac{i}{\sigma} (E - W(q)) - \frac{1}{2} f'(q) \right) \psi(q) \quad W(q) = V(q) - f(q)g(q)$$



Solution: 
$$\psi(q) = \frac{\psi_0}{\sqrt{f(q)}} e^{\frac{i}{\sigma} \left( \varphi + E \int \frac{dq}{f(q)} - \int \frac{W(q)}{f(q)} dq \right)}$$

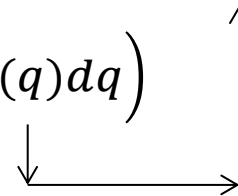
With the Dirac consistency equation:

$$f(q) V''(q) = 0 \longrightarrow W(q) = k q - f(q) g(q)$$



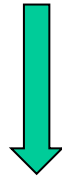
Solution undetermined by the  $g(q)$  function

$$\psi(q) = \frac{\psi_0}{\sqrt{f(q)}} e^{\frac{i}{\sigma} \left( \varphi + E \int \frac{dq}{f(q)} - k \int \frac{q}{f(q)} dq + \int g(q) dq \right)}$$



## 5. Application case: the logistic function.

$$\psi(q) = \frac{\psi_0}{\sqrt{f(q)}} e^{\frac{i}{\sigma} \left( \varphi + E \int \frac{dq}{f(q)} - k \int \frac{q}{f(q)} dq + \int g(q) dq \right)}$$



$$\longleftarrow f(q) = a q - b q^2 = q(a - b q)$$

$$\psi(q) = \begin{cases} \frac{\psi_0}{\sqrt{-q(a - b q)}} e^{-\frac{\pi E}{\sigma a} + \frac{i}{\sigma} \left( \frac{\pi}{2} + \varphi + \frac{E}{a} \ln(-q) + \left( \frac{k}{b} - \frac{E}{a} \right) \ln(a - b q) + \int g(q) dq \right)} : q < 0 \\ \frac{\psi_0}{\sqrt{q(a - b q)}} e^{\frac{i}{\sigma} \left( \varphi + \frac{E}{a} \ln q + \left( \frac{k}{b} - \frac{E}{a} \right) \ln(a - b q) + \int g(q) dq \right)} : 0 < q < \frac{a}{b} \\ \frac{\psi_0}{\sqrt{q(b q - a)}} e^{-\frac{\pi}{\sigma} \left( \frac{k}{b} - \frac{E}{a} \right) + \frac{i}{\sigma} \left( \frac{\pi}{2} + \varphi + \frac{E}{a} \ln q + \left( \frac{k}{b} - \frac{E}{a} \right) \ln(b q - a) + \int g(q) dq \right)} : q > \frac{a}{b} \end{cases}$$

The research is centered in finding the mathematical pattern of the  $g(q)$  function that permits to undo the singularities and that provides the quantization of the system Energy  $E$ .