

AN ANALYTICAL FORMALISM OF DYNAMICAL SYSTEMS


Symposium 11: Mathematical modeling of complex systems

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1. Dynamical systems: mechanical approach versus teleological approach.
2. The Lagrangian approach: the Havas approach.
3. The Hamiltonian approach: the Dirac approach.
4. Application to the one-dimensional autonomous systems.
5. Thermodynamics interpretation of the analytical formalism.

1. Dynamical systems: mechanical approach versus teleological approach.

Dynamical system: a system of coupled first order differential equations


$$\dot{q}_i(t) = f_i(t, \mathbf{q}) \quad i=1, 2, \dots, n.$$

Mechanical approach: mathematical representation of the causal relations among the system variables.

Teleological approach: can the system follow a trajectory given by an objective? → Yes: the system trajectory is that one that minimizes the **Action S(t)**:

$$S(t) = \int_{t_1}^{t_2} L(t, \mathbf{q}, \dot{\mathbf{q}}) dt \quad L(t, \mathbf{q}, \dot{\mathbf{q}}): \text{Lagrangian function}$$

2. The Lagrangian approach: the Havas approach.

$$S(t) = \int_{t_1}^{t_2} L(t, \mathbf{q}, \dot{\mathbf{q}}) dt \quad \overset{i?}{\longleftrightarrow} \quad \dot{q}_i(t) = f_i(t, \mathbf{q}) \quad i=1, 2, \dots, n.$$

How to become equivalent both formulations?

If $S(t)$ must be minimum $\longrightarrow \delta S(t) = 0$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad j=1, 2, \dots, n.$$

Lagrange Equations

$\dot{q}_i(t) = f_i(t, \mathbf{q}) \quad j=1, 2, \dots, n.$ This is the so-called as:
Lagrange inverse problem

2. The Lagrangian approach: the Havas approach.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$$

Lagrange Equations

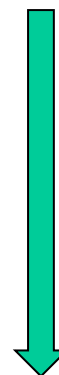
Lagrange inverse problem



$$\dot{q}_i(t) = f_i(t, \mathbf{q})$$

Differential system

P. Havas solution (1973)



$$L(t, \mathbf{q}, \dot{\mathbf{q}}) = g_k(t, \mathbf{q}) \dot{q}_k - V(t, \mathbf{q})$$



$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$$

$$F_{ij}(t, \mathbf{q}) \dot{q}_j = - \frac{\partial g_i(t, \mathbf{q})}{\partial t} - \frac{\partial V(t, \mathbf{q})}{\partial q_i}$$

Vector field equations

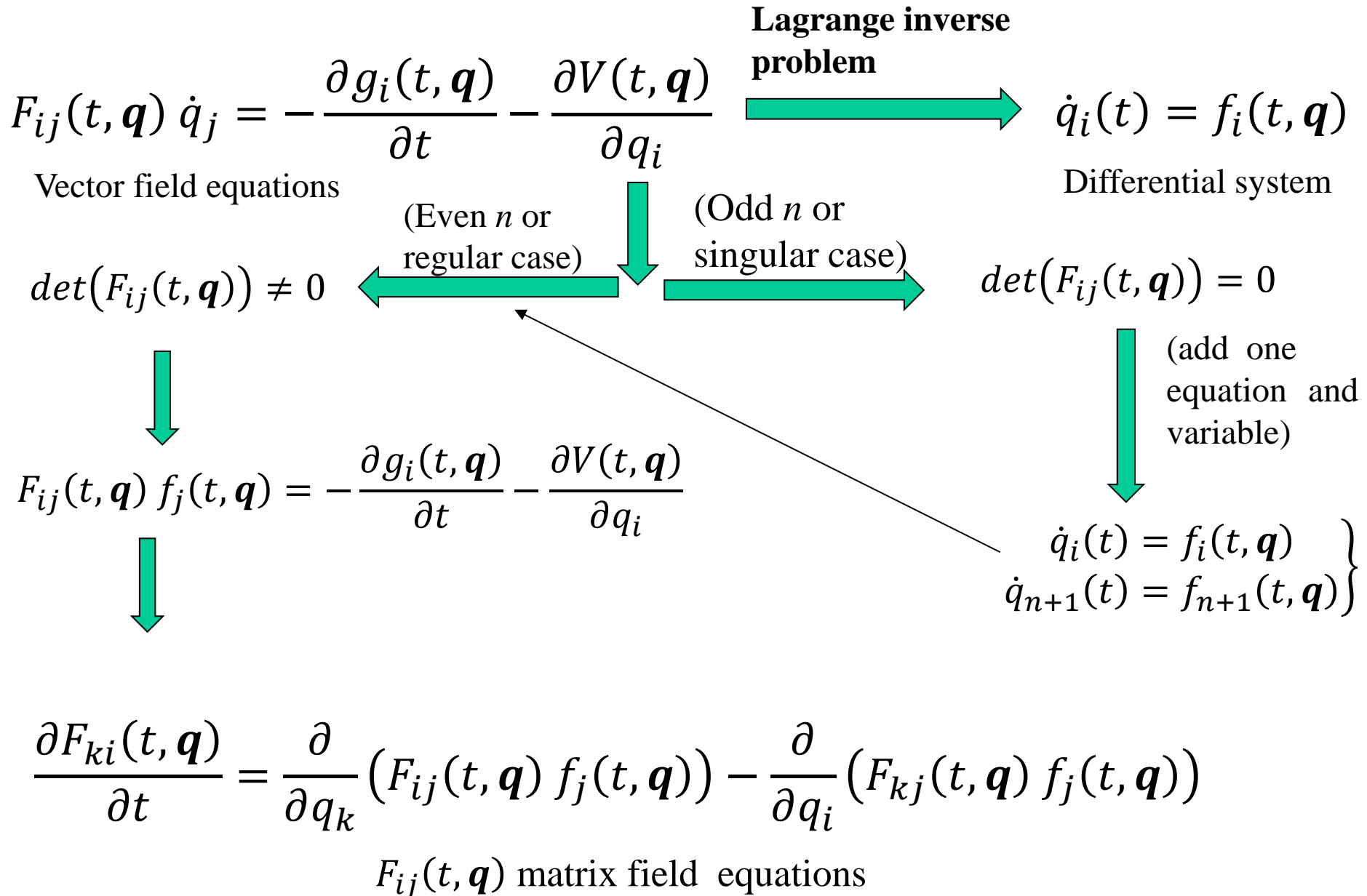
such that:

$$F_{ij}(t, \mathbf{q}) = \frac{\partial g_i(t, \mathbf{q})}{\partial q_j} - \frac{\partial g_j(t, \mathbf{q})}{\partial q_i}$$

F_{ij} antisymmetric matrix

(convention of the repeated subscripts assumed as a sum from now onwards)

2. The Lagrangian approach: the Havas approach.



2. The Lagrangian approach: the Havas approach.

Summarizing the **Havas approach**:

1. Add a new equation $\dot{q}_{n+1}(t) = f_{n+1}(t, \mathbf{q}, q_{n+1})$ to (1) if the dimension n is odd.

2. Solve the $F_{ij}(t, \mathbf{q})$ functions by:

$$\frac{\partial F_{ki}(t, \mathbf{q})}{\partial t} = \frac{\partial}{\partial q_k} (F_{ij}(t, \mathbf{q}) f_j(t, \mathbf{q})) - \frac{\partial}{\partial q_i} (F_{kj}(t, \mathbf{q}) f_j(t, \mathbf{q}))$$

3. Solve the $g_i(t, \mathbf{q})$ functions by: $F_{ij}(t, \mathbf{q}) = \frac{\partial g_i(t, \mathbf{q})}{\partial q_j} - \frac{\partial g_i(t, \mathbf{q})}{\partial q_j}$

4. Solve the $V(t, \mathbf{q})$ function by: $F_{ij}(t, \mathbf{q}) \dot{q}_j = -\frac{\partial g_i(t, \mathbf{q})}{\partial t} - \frac{\partial V(t, \mathbf{q})}{\partial q_i}$

5. The Lagrangian is given by: $L(t, \mathbf{q}, \dot{\mathbf{q}}) = g_k(t, \mathbf{q}) \dot{q}_k - V(t, \mathbf{q})$

Note: in those cases where the F_{ij} , g_i and V functions do not depend explicitly on the time, the steps 3 and 4 can be exchanged.

3. The Hamiltonian approach: the Dirac approach.

Momenta

$$p_j = \frac{\partial L(t, \mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{q}_j} \quad \longrightarrow \quad \text{Hamiltonian function}$$

$$H(t, \mathbf{q}, \mathbf{p}) = p_j \dot{q}_j - L(t, \mathbf{q}, \dot{\mathbf{q}})$$

Hamilton equations:
equivalent to Lagrange equations

$$\left. \begin{aligned} \dot{q}_j &= \frac{\partial H(t, \mathbf{q}, \mathbf{p})}{\partial p_j} \\ \dot{p}_j &= - \frac{\partial H(t, \mathbf{q}, \mathbf{p})}{\partial q_j} \end{aligned} \right\}$$

Present case: $L(t, \mathbf{q}, \dot{\mathbf{q}}) = g_k(t, \mathbf{q}) \dot{q}_k - V(t, \mathbf{q})$

$$\downarrow \longleftarrow p_j = g_j(t, \mathbf{q})$$

$$H(t, \mathbf{q}, \mathbf{p}) = V(t, \mathbf{q})$$

Problem!! \longrightarrow The Hamiltonian do not depend on momenta

3. The Hamiltonian approach: the Dirac approach.

The **Dirac approach** (Paul A. M. Dirac, 1964):

1. Define the **primary constraints**: $\phi_m(t, \mathbf{q}, \mathbf{p}) = p_m - g_m(t, \mathbf{q}) = 0$
2. Insert them in the Hamiltonian:

$$H_T(t, \mathbf{q}, \mathbf{p}) = V(t, \mathbf{q}) + \lambda_m(t, \mathbf{q}, \mathbf{p}) \phi_m(t, \mathbf{q}, \mathbf{p})$$

$\lambda_m(t, \mathbf{q}, \mathbf{p})$  unknown multiplying functions to be found

3. Hamilton equations for H_T :

$$\left. \begin{aligned} \dot{q}_j &= \frac{\partial H_T(t, \mathbf{q}, \mathbf{p})}{\partial p_j} = \lambda_j(t, \mathbf{q}, \mathbf{p}) \\ \dot{p}_j &= -\frac{\partial H_T(t, \mathbf{q}, \mathbf{p})}{\partial q_j} = -\frac{\partial V(t, \mathbf{q})}{\partial q_j} + \lambda_m(t, \mathbf{q}, \mathbf{p}) \frac{\partial g_m(t, \mathbf{q})}{\partial q_j} \end{aligned} \right\}$$

3. The Hamiltonian approach: the Dirac approach.

4. State the **consistency conditions** for the primary constants ϕ_m :

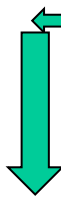
$$\dot{\phi}_l(t, \mathbf{q}, \mathbf{p}) = \frac{\partial \phi_l(t, \mathbf{q}, \mathbf{p})}{\partial t} + \frac{\partial \phi_l(t, \mathbf{q}, \mathbf{p})}{\partial q_k} \dot{q}_k + \frac{\partial \phi_l(t, \mathbf{q}, \mathbf{p})}{\partial p_k} \dot{p}_k = 0$$



← Hamilton equations

$$F_{lm}(t, \mathbf{q}) \lambda_m(t, \mathbf{q}, \mathbf{p}) = - \frac{\partial g_l(t, \mathbf{q})}{\partial t} - \frac{\partial V(t, \mathbf{q})}{\partial q_l}$$

Follow
Havas
approach



$$\det(F_{ij}(t, \mathbf{q})) \neq 0$$



$$\lambda_m(t, \mathbf{q}, \mathbf{p}) = f_m(t, \mathbf{q})$$



No approach followed



Even n



Odd n

$$\lambda_m(t, \mathbf{q}, \mathbf{p}) = f_m(t, \mathbf{q})$$

But some either some $g_l(t, \mathbf{q})$ functions or $V(t, \mathbf{q})$ function are undetermined

3. The Hamiltonian approach: the Dirac approach.

5. Hamiltonian: $H_T(t, \mathbf{q}, \mathbf{p})$
 $= V(t, \mathbf{q}) + f_m(t, \mathbf{q})(p_m - g_m(t, \mathbf{q}))$

6. If $g_l(\mathbf{q})$ and $V(\mathbf{q})$ independent on time and the system is autonomous, i.e., $f_m(\mathbf{q})$, the system **Energy E** can be stated as a motion constant:

$$E = V(\mathbf{q}) + f_m(\mathbf{q})(p_m - g_m(\mathbf{q}))$$

7. Hamilton-Jacobi equation: $S(t, \mathbf{q}) \longrightarrow$ Action

$$\frac{\partial S(t, \mathbf{q})}{\partial t} + f_m(t, \mathbf{q}) \left(\frac{\partial S(t, \mathbf{q})}{\partial q_m} - g_m(t, \mathbf{q}) \right) + V(t, \mathbf{q}) = 0$$

This equation is very important to interpret the quantum approach

4. Application to the one-dimensional autonomous systems.

$$\dot{q}(t) = f(t, q) \quad \xrightarrow{\text{Autonomous}} \quad \dot{q}(t) = f(q)$$

Havas approach:

$$\left. \begin{array}{l} \dot{q}_1(t) = f_1(q_1) \\ \dot{q}_2(t) = f_2(q_1) \end{array} \right\} \begin{array}{l} \longrightarrow \dot{q}(t) = f(q) \\ \longrightarrow \text{Arbitrary } f_2 \end{array}$$

From the simplifying hypotheses that $g_l(\mathbf{q})$ and $V(\mathbf{q})$ are independent on time :

$$L(\mathbf{q}) = \frac{e^{q_2 - \int \frac{f_2(q_1)}{f_1(q_1)} dq_1}}{f_1(q_1)} \dot{q}_1 + e^{q_2} \dot{q}_2 - e^{q_2 - \int \frac{f_2(q_1)}{f_1(q_1)} dq_1}$$

$$E = H(\mathbf{q}, \mathbf{p}) = f_1(q_1) p_1 + f_2(q_1) p_2 - f_2(q_1) e^{q_2}$$



$$\longleftarrow \dot{q}_2(t) = f_2(q_1) = 1, \text{ i.e., } q_2 = t$$

$$E = H(\mathbf{q}, \mathbf{p}) = f_1(q_1) p_1 + p_2 - e^{q_2}$$

4. Application to the one-dimensional autonomous systems.

Dirac approach to: $\dot{q}(t) = f(t, q)$

$$H_T(t, q, p) = V(t, q) + \lambda(t, q, p) \phi(t, q, p) \quad \phi(t, q, p) = p - g(t, q)$$

Consistency condition: $\dot{\phi} = \frac{\partial g(t, q)}{\partial t} + \frac{\partial V(t, q)}{\partial q} = 0$ Problem: $\lambda(t, q, p)$ is not present



Secondary constraint: $\chi(t, q, p) = \frac{\partial g(t, q)}{\partial t} + \frac{\partial V(t, q)}{\partial q} = 0$



Consistency condition: $\dot{\chi}(t, q, p) = 0$



$$\frac{\partial^2 g(t, q)}{\partial t^2} + \frac{\partial^2 V(t, q)}{\partial t \partial q} + f(t, q) \left(\frac{\partial^2 g(t, q)}{\partial q \partial t} + \frac{\partial^2 V(t, q)}{\partial q^2} \right) = 0$$

4. Application to the one-dimensional autonomous systems.

$$\dot{q}(t) = f(t, q) \quad \xrightarrow{\text{Autonomous}} \quad \dot{q}(t) = f(q)$$

From the simplifying hypotheses that $g_l(\mathbf{q})$ and $V(\mathbf{q})$ are independent on time, the consistency condition becomes:

$$f(q) V''(q) = 0$$



$$E = H_T(q, p) = f(q) p + k q - f(q) g(q)$$

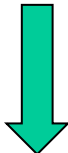
The **system energy** is undetermined by the $g(q)$ function

5. Thermodynamics interpretation of the analytical formalism.

Consider the autonomous system: $\dot{q}_i(t) = f_i(\mathbf{q})$

The Hamiltonian is:

$$H_T(t, \mathbf{q}, \mathbf{p}) = \sum_k f_k(\mathbf{q}) p_k + V(t, \mathbf{q}) - \sum_k f_k(\mathbf{q}) g_k(t, \mathbf{q})$$

Develop $f_k(\mathbf{q})$ about an attractor \mathbf{q}_a  $f_k(\mathbf{q}) = \sum_i A_{ki}(q_i - q_{ia}) + F_k(\mathbf{q})$

$$H_T(t, \mathbf{q}, \mathbf{p}) = \sum_k A_{kk} q_k p_k + \sum_{k \neq i} A_{ki} q_i p_k - \sum_{ki} A_{ki} q_{ia} p_k + \sum_k F_k(\mathbf{q}) p_k + V(t, \mathbf{q}) - \sum_k f_k(\mathbf{q}) g_k(t, \mathbf{q})$$

Identify H_T with the thermodynamic Euler equation:

$$U = T \Omega + Y_k X_k$$

T the temperature and Ω the entropy

Y_k as extensive variables and X_k as intensive variables

Identification: $Y_k \rightarrow A_{kk} p_k$ and $X_k \rightarrow q_k$

Identification: $T\Omega = \sum_{k \neq i} A_{ki} p_k q_i - \sum_{ki} A_{ki} q_{ia} p_k + \sum_k F_k(\mathbf{q}) p_k + V(t, \mathbf{q}) - \sum_k f_k(\mathbf{q}) g_k(t, \mathbf{q})$