(Non)contextuality and its applications

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- Systems of random variables: contents, contexts, and stochastic relations.

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- Application to cyclic systems

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- Contextuality
- Measures of contextuality and noncontextuality
- Application to cyclic systems
- Epistemic/Bayesian random variables: Liar's paradox, M. C. Escher's picture, impossible figures.

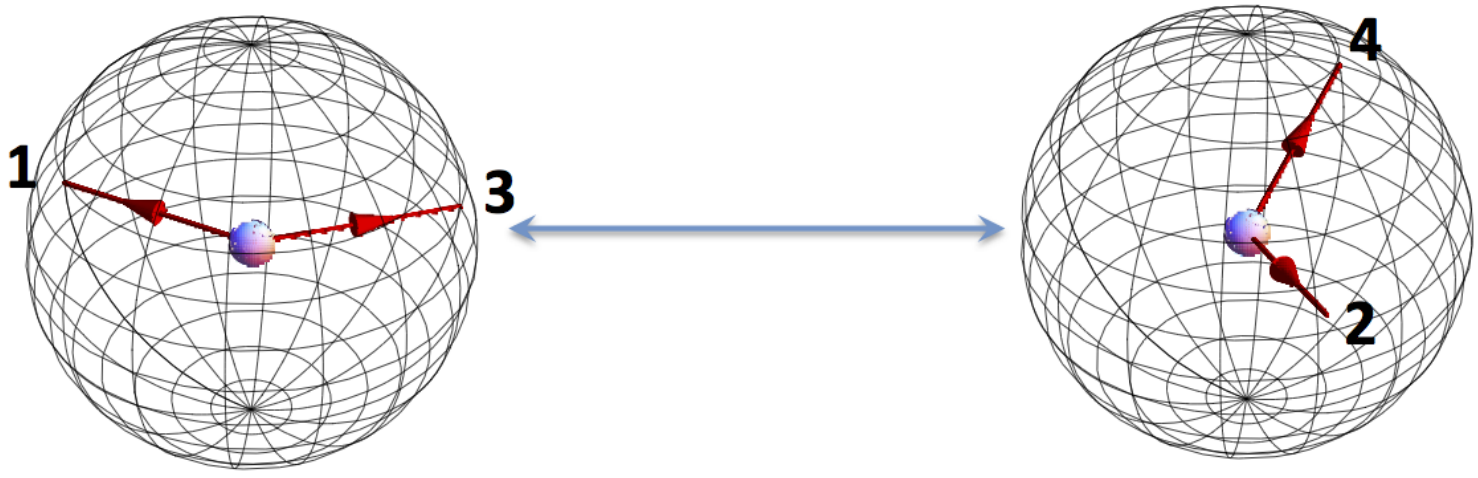
SYSTEM OF RANDOM VARIABLES

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

q_1, q_2, q_3, q_4, q_5 — contents (measurements, questions)

c^1, c^2, c^3, c^4— contexts, or conditions

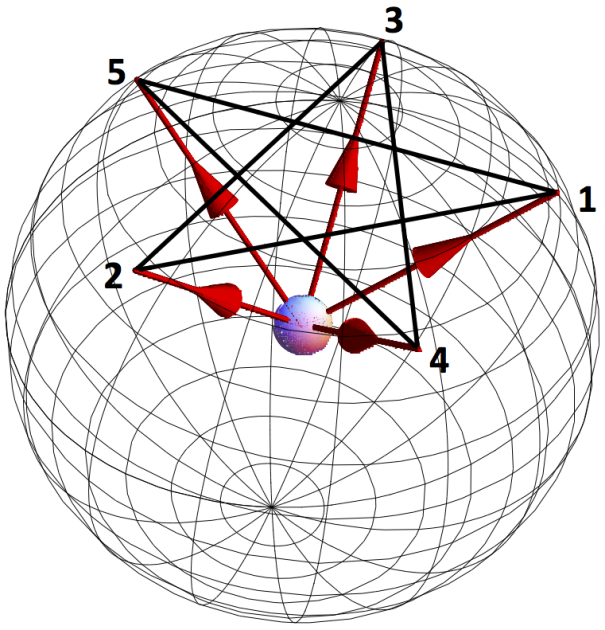
EPR/B SYSTEM



R_1^1	R_2^1			c^1
	R_2^2	R_3^2		c^2
		R_3^3	R_4^3	c^3
R_1^4			R_4^4	c^4
q_1	q_2	q_3	q_4	$\mathcal{R}_4$

q_i = is the spin along axis i "up"?

KCBS SYSTEM



R_1^1	R_2^1				c^1
	R_2^2	R_3^2			c^2
		R_3^3	R_4^3		c^3
			R_4^4	R_5^4	c^4
R_1^5				R_5^5	c^5
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}_5$

q_i = is the spin along axis i "up"?

SYSTEM OF RANDOM VARIABLES

R_1^1	R_2^1	R_3^1	c^1
R_1^2	R_2^2	R_3^2	c^2
R_1^3	R_2^3	R_3^3	c^3
q_1	q_2	q_3	$\mathcal{X}$

R_1^1	R_2^1	c^1
R_1^2	R_2^2	c^2
q_1	q_2	$\mathcal{R}_2$

context is not just a subset of contents/questions

SYSTEM OF HUMAN RESPONSES TO QUESTIONS

R_1^1	R_2^1	$c^1 = q_1 \to q_2$
R_1^2	R_2^2	$c^2 = q_2 \to q_1$
$q_1 =$ "overseas trip?"	$q_2 =$ "coronavirus?"	$\mathcal{R}_2$

SYSTEM OF HUMAN RESPONSES TO QUESTIONS

	$R_2^1 = +1$	$R_2^1 = -1$	
$R_1^1 = +1$			$1/2$
$R_1^1 = -1$			$1/2$
	$1/2$	$1/2$	

R_1^1	R_2^1	$c^1 = q_1 \to q_2$
R_1^2	R_2^2	$c^2 = q_2 \to q_1$
$q_1 =$ "overseas trip?"	$q_2 =$ "coronavirus?"	$\mathcal{R}_2$

	$R_2^2 = +1$	$R_2^2 = -1$	
$R_1^2 = +1$			$1/2$
$R_1^2 = -1$			$1/2$
	$1/2$	$1/2$	

SYSTEM OF HUMAN RESPONSES TO QUESTIONS

$R_1^1 = +1$		$1/2$
$R_1^1 = -1$		$1/2$

R_1^1		$c^1 = q_1 \to q_2$
R_1^2		$c^2 = q_2 \to q_1$
$q_1 =$ "overseas trip?"		$\mathcal{R}_2$

$R_1^2 = +1$		$1/2$
$R_1^2 = -1$		$1/2$

SYSTEM OF HUMAN RESPONSES TO QUESTIONS

$R_1 = +1$		$1/2$
$R_1 = -1$		$1/2$

R_1		
$q_1 =$ "overseas trip?"		

SYSTEM OF HUMAN RESPONSES TO QUESTIONS

	$R_2^1 = +1$	$R_2^1 = -1$	
	$1/2$	$1/2$	

	R_2^1	$c^1 = q_1 \to q_2$
	R_2^2	$c^2 = q_2 \to q_1$
	$q_2 =$ "coronavirus?"	$\mathcal{R}_2$

	$R_2^2 = +1$	$R_2^2 = -1$	
	$1/2$	$1/2$	

SYSTEM OF HUMAN RESPONSES TO QUESTIONS

	R_2	
	$q_2 =$ "coronavirus?"	

	$R_2 = +1$	$R_2 = -1$	
	$1/2$	$1/2$	

CONSISTENTLY CONNECTED SYSTEM

	$R_2^1 = +1$	$R_2^1 = -1$	
$R_1^1 = +1$			$1/2$
$R_1^1 = -1$			$1/2$
	$1/2$	$1/2$	

R_1^1	R_2^1	$c^1 = q_1 \to q_2$
R_1^2	R_2^2	$c^2 = q_2 \to q_1$
$q_1 =$ "overseas trip?"	$q_2 =$ "coronavirus?"	$\mathcal{R}_2$

	$R_2^2 = +1$	$R_2^2 = -1$	
$R_1^2 = +1$			$1/2$
$R_1^2 = -1$			$1/2$
	$1/2$	$1/2$	

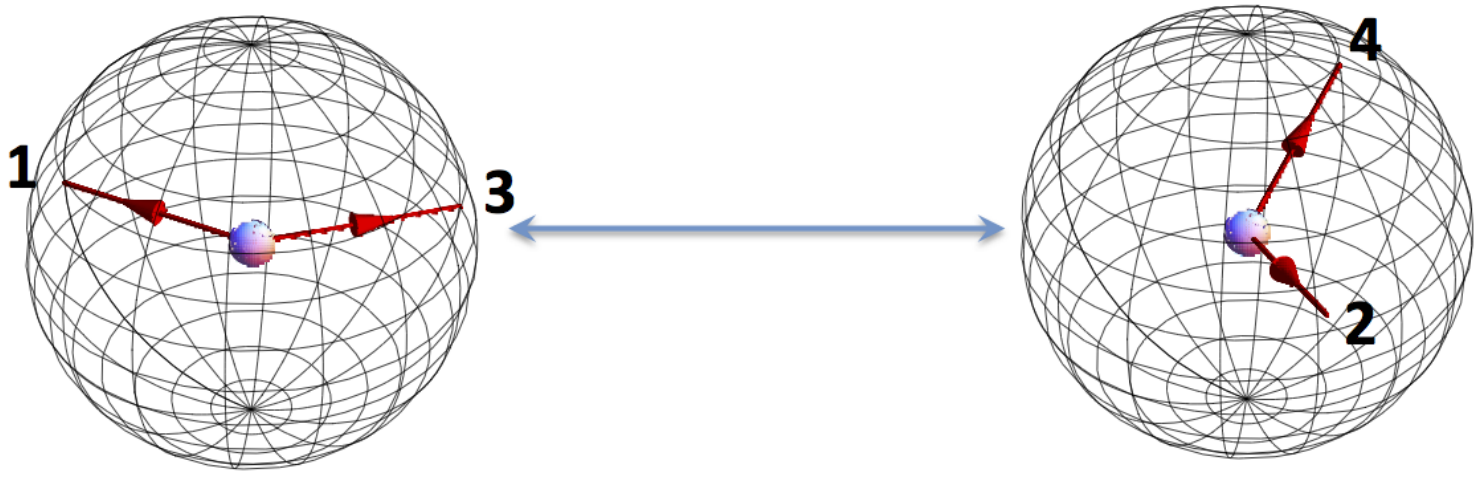
CONTEXTUAL SYSTEM

	$R_2^1 = +1$	$R_2^1 = -1$	
$R_1^1 = +1$	0	$1/2$	$1/2$
$R_1^1 = -1$	$1/2$	0	$1/2$
	$1/2$	$1/2$	

R_1^1	R_2^1	$c^1 = q_1 \to q_2$
R_1^2	R_2^2	$c^2 = q_2 \to q_1$
$q_1 =$ "overseas trip?"	$q_2 =$ "coronavirus?"	$\mathcal{R}_2$

	$R_2^2 = +1$	$R_2^2 = -1$	
$R_1^2 = +1$	$1/4$	$1/4$	$1/2$
$R_1^2 = -1$	$1/4$	$1/4$	$1/2$
	$1/2$	$1/2$	

EPR/B SYSTEM



R_1^1	R_2^1			c^1
	R_2^2	R_3^2		c^2
		R_3^3	R_4^3	c^3
R_1^4			R_4^4	c^4
q_1	q_2	q_3	q_4	$\mathcal{R}_4$

q_i = is the spin along axis i "up"?

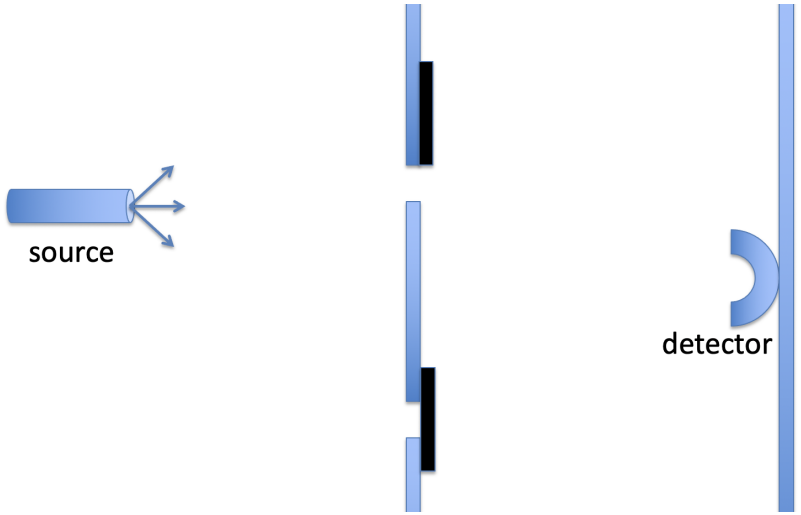
INCONSISTENTLY CONNECTED SYSTEM ("QUESTION ORDER EFFECT")

R_1^1	R_2^1	$c^1 = q_1 \to q_2$
R_1^2	R_2^2	$c^2 = q_2 \to q_1$
$q_1 =$ "overseas trip?"	$q_2 =$ "coronavirus?"	$\mathcal{R}_2$

INCONSISTENTLY CONNECTED SYSTEM ("SIGNALING IN TIME")

R_1^1	R_2^1		c^1
	R_2^2	R_3^2	c^2
R_1^3		R_3^3	c^3
q_1	q_2	q_3	$\mathcal{R}_3$

INCONSISTENTLY CONNECTED SYSTEM (TWO-SLIT EXPERIMENT)



$R^{\circ\circ}_{\circ\cdot}$	$R^{\circ\circ}_{\cdot\circ}$			$c_{\circ\circ}$
	$R^{\times\circ}_{\cdot\circ}$	$R^{\times\circ}_{\times\cdot}$		$c_{\times\circ}$
		$R^{\times\times}_{\times\cdot}$	$R^{\times\times}_{\cdot\times}$	$c_{\times\times}$
$R^{\circ\times}_{\circ\cdot}$			$R^{\circ\times}_{\cdot\times}$	$c_{\circ\times}$
$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$	$\mathcal{R}_4$

$q_{\circ\cdot}$: hit through left open slit?
$q_{\cdot\circ}$: hit through right open slit?
$q_{\times\cdot}$: hit through left closed slit?
$q_{\cdot\times}$: hit through right closed slit?

INCONSISTENTLY CONNECTED SYSTEM

R_1^1	R_2^1	$c^1 = q_1 \rightarrow q_2$
R_1^2	R_2^2	$c^2 = q_2 \rightarrow q_1$
$q_1 =$ "overseas trip?"	$q_2 =$ "coronavirus?"	$\mathcal{R}_2$

HOW DIFFERENT ARE THEY?

R_1^1		$c^1 = q_1 \rightarrow q_2$
R_1^2		$c^2 = q_2 \rightarrow q_1$
$q_1 =$ "overseas trip?"		

HOW DIFFERENT ARE THEY?

R_1^1		$c^1 = q_1 \rightarrow q_2$
R_1^2		$c^2 = q_2 \rightarrow q_1$
$q_1 =$ "overseas trip?"		

	$R_1^1 = +1$	$R_1^1 = -1$
$R_1^2 = +1$		q
$R_1^2 = -1$	p	$1 - q$
	$1 - p$	

HOW DIFFERENT ARE THEY?

S_1^1		
S_1^2		

	$S_1^1 = +1$	$S_1^1 = -1$
$S_1^2 = +1$		q
$S_1^2 = -1$	p	$1 - q$
	p	$1 - p$

HOW DIFFERENT ARE THEY? COUPLINGS

S_1^1		
S_1^2		

	$S_1^1 = +1$	$S_1^1 = -1$	
$S_1^2 = +1$			q
$S_1^2 = -1$			$1 - q$
	p	$1 - p$	

HOW DIFFERENT ARE THEY? MAXIMAL COUPLING

S_1^1		
S_1^2		

	$S_1^1 = +1$	$S_1^1 = -1$	
$S_1^2 = +1$	$\min(p, q)$	$q - \min(p, q)$	q
$S_1^2 = -1$	$p - \min(p, q)$	$1 - p - q + \min(p, q)$	$1 - q$
	p	$1 - p$	

HOW DIFFERENT ARE THEY? MAXIMAL COUPLING

S_1^1		
S_1^2		

	$S_1^1 = +1$	$S_1^1 = -1$	
$S_1^2 = +1$	$\min(p, q)$	$q - \min(p, q)$	q
$S_1^2 = -1$	$p - \min(p, q)$	$1 - p - q + \min(p, q)$	$1 - q$
	p	$1 - p$	

$$\Pr\left[S_1^1 \neq S_1^2\right] = |p - q|$$

HOW DIFFERENT ARE THEY? MAXIMAL COUPLING

S_1^1		
S_1^2		

	$S_1^1 = +1$	$S_1^1 = -1$	
$S_1^2 = +1$	$\min(p, q)$	$q - \min(p, q)$	q
$S_1^2 = -1$	$p - \min(p, q)$	$1 - p - q + \min(p, q)$	$1 - q$
	p	$1 - p$	

$$\Pr\left[S_1^1 \neq S_1^2\right] = |p - q| = \mathsf{TotVarDis}\left[R_1^1, R_1^2\right]$$

DEFINITION OF (NON)CONTEXTUALITY

R_1^1	R_2^1	$c^1 = q_1 \to q_2$
R_1^2	R_2^2	$c^2 = q_2 \to q_1$
$q_1 =$ "overseas trip?"	$q_2 =$ "coronavirus?"	$\mathcal{R}_2$

S_1^1	S_2^1	
S_1^2	S_2^2	

$$\Pr\left[S_1^1 \neq S_1^2\right] = \text{TotVarDis}\left[R_1^1, R_1^2\right]$$

and

$$\Pr\left[S_2^1 \neq S_2^2\right] = \text{TotVarDis}\left[R_2^1, R_2^2\right]$$

DEFINITION OF (NON)CONTEXTUALITY

S_1^1	S_2^1	
S_1^2	S_2^2	

?

S_1^1	S_2^1	$c^1 = q_1 \to q_2$
S_1^2	S_2^2	$c^2 = q_2 \to q_1$
$q_1 =$ "overseas trip?"	$q_2 =$ "coronavirus?"	S_2

?

?

$$\Pr\left[S_1^1 \neq S_1^2\right] = \text{TotVarDis}\left[R_1^1, R_1^2\right]$$

and

$$\Pr\left[S_2^1 \neq S_2^2\right] = \text{TotVarDis}\left[R_2^1, R_2^2\right]$$

?

DEFINITION OF (NON)CONTEXTUALITY

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

DEFINITION OF (NON)CONTEXTUALITY

s_1^1	s_2^1		s_4^1		
s_1^2		s_3^2			
	s_2^3	s_3^3	s_4^3	s_5^3	
		s_3^4		s_5^4	

DEFINITION OF (NON)CONTEXTUALITY

s_1^1	s_2^1		s_4^1		
s_1^2		s_3^2			
	s_2^3	s_3^3	s_4^3	s_5^3	
		s_3^4		s_5^4	

DEFINITION OF (NON)CONTEXTUALITY

s_1^1	s_2^1		s_4^1		
s_1^2		s_3^2			
	s_2^3	s_3^3	s_4^3	s_5^3	
		s_3^4		s_5^4	

DEFINITION OF (NON)CONTEXTUALITY

s_1^1	s_2^1		s_4^1		
s_1^2		s_3^2			
	s_2^3	s_3^3	s_4^3	s_5^3	
		s_3^4		s_5^4	

DEFINITION OF (NON)CONTEXTUALITY

Definition

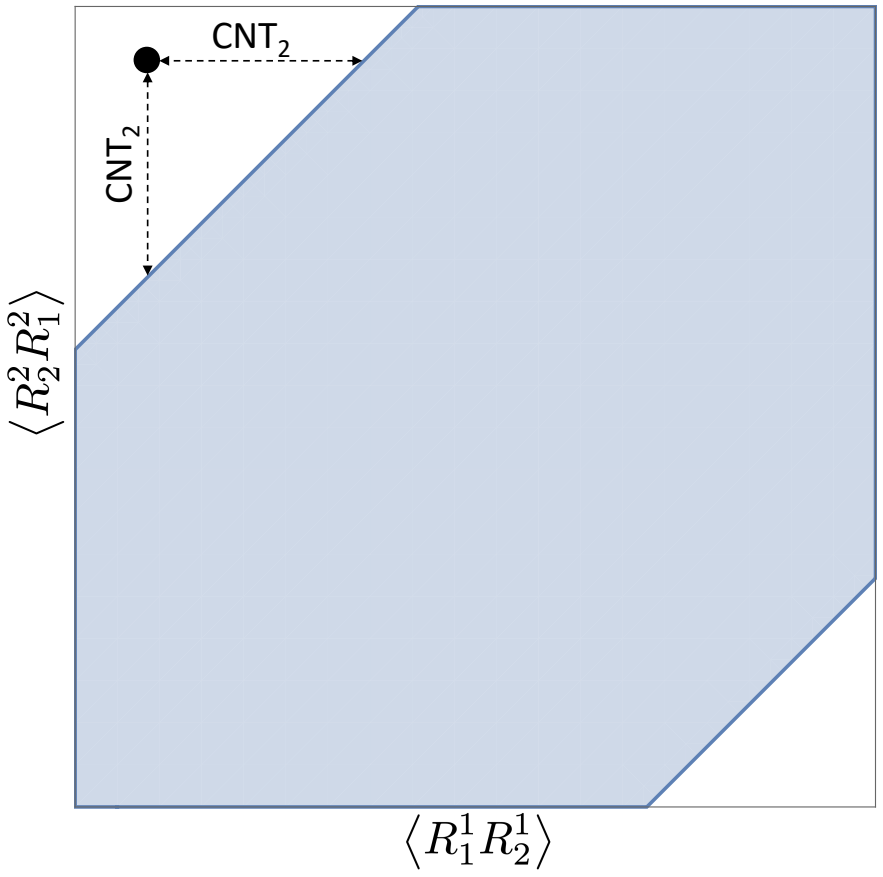
A system is noncontextual if it has a coupling in which any two random variables sharing a content (answering the same question) coincide with the maximal possible probability.

If no such coupling exists, the system is contextual: the contexts are "forcing" the variables to be more dissimilar than they are when taken in isolation.

MEASURING (NON)CONTEXTUALITY

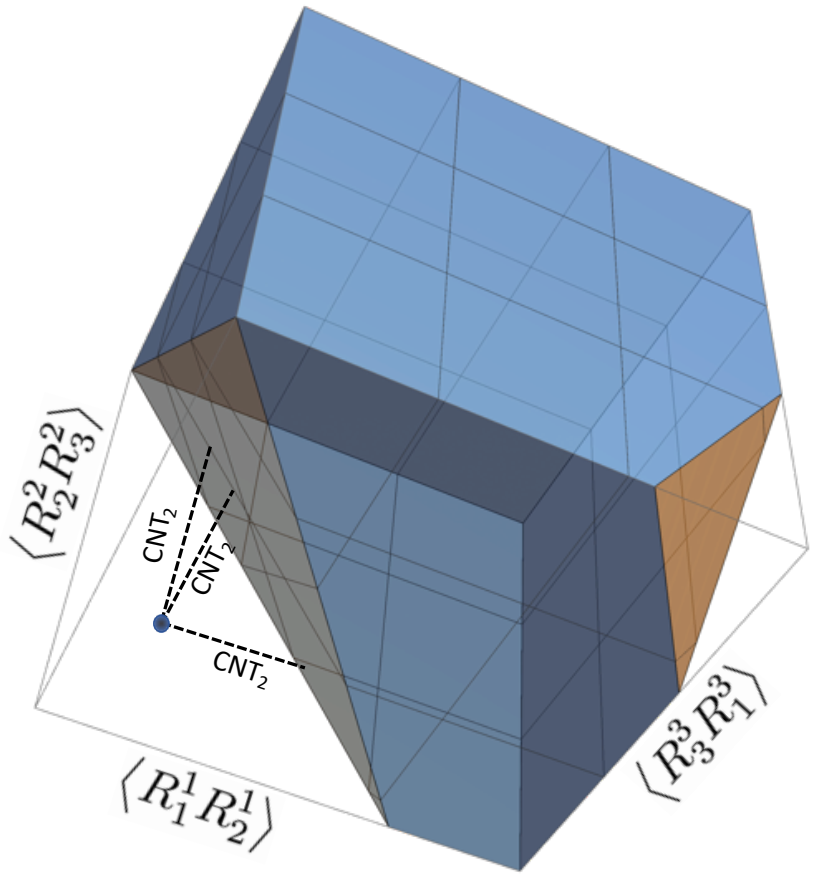
DEGREE OF CONTEXTUALITY IN

R_1^1	R_2^1
R_1^2	R_2^2

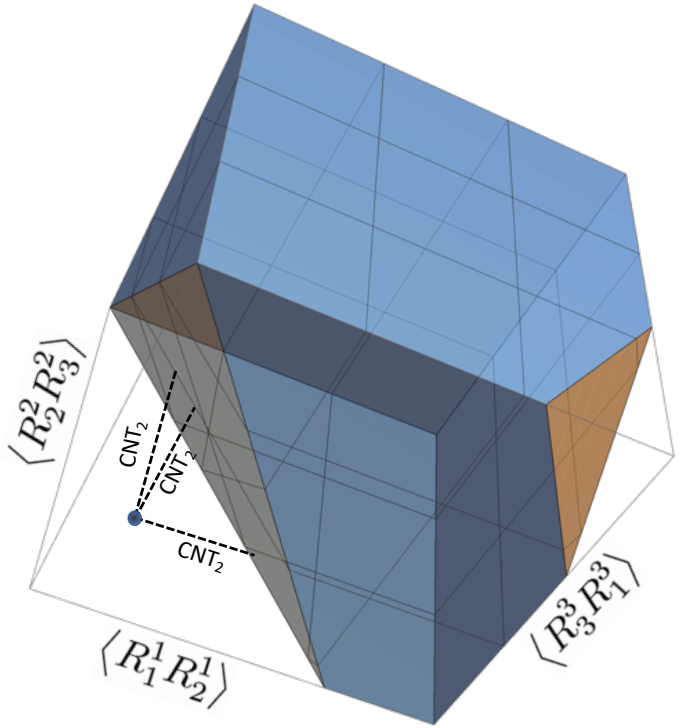


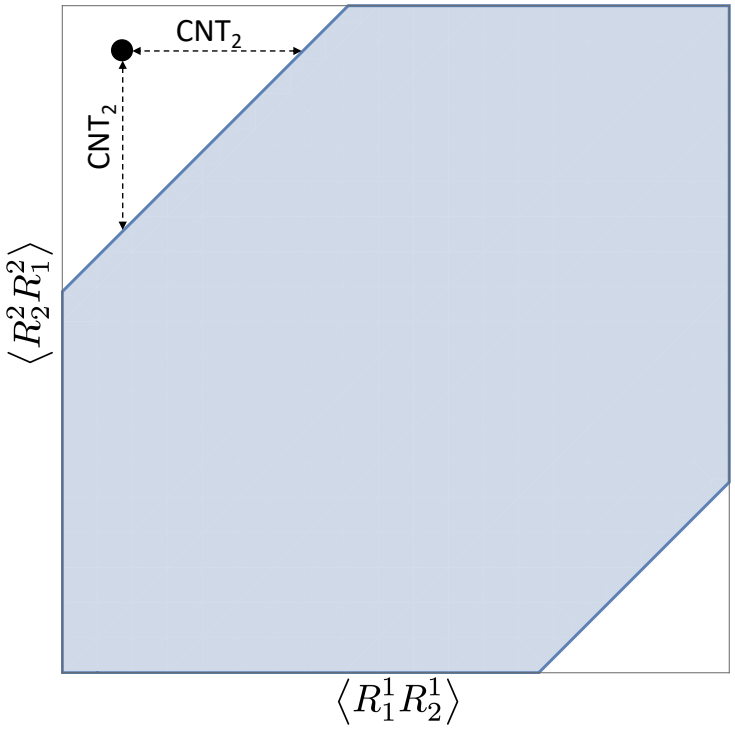
DEGREE OF CONTEXTUALITY IN

R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3

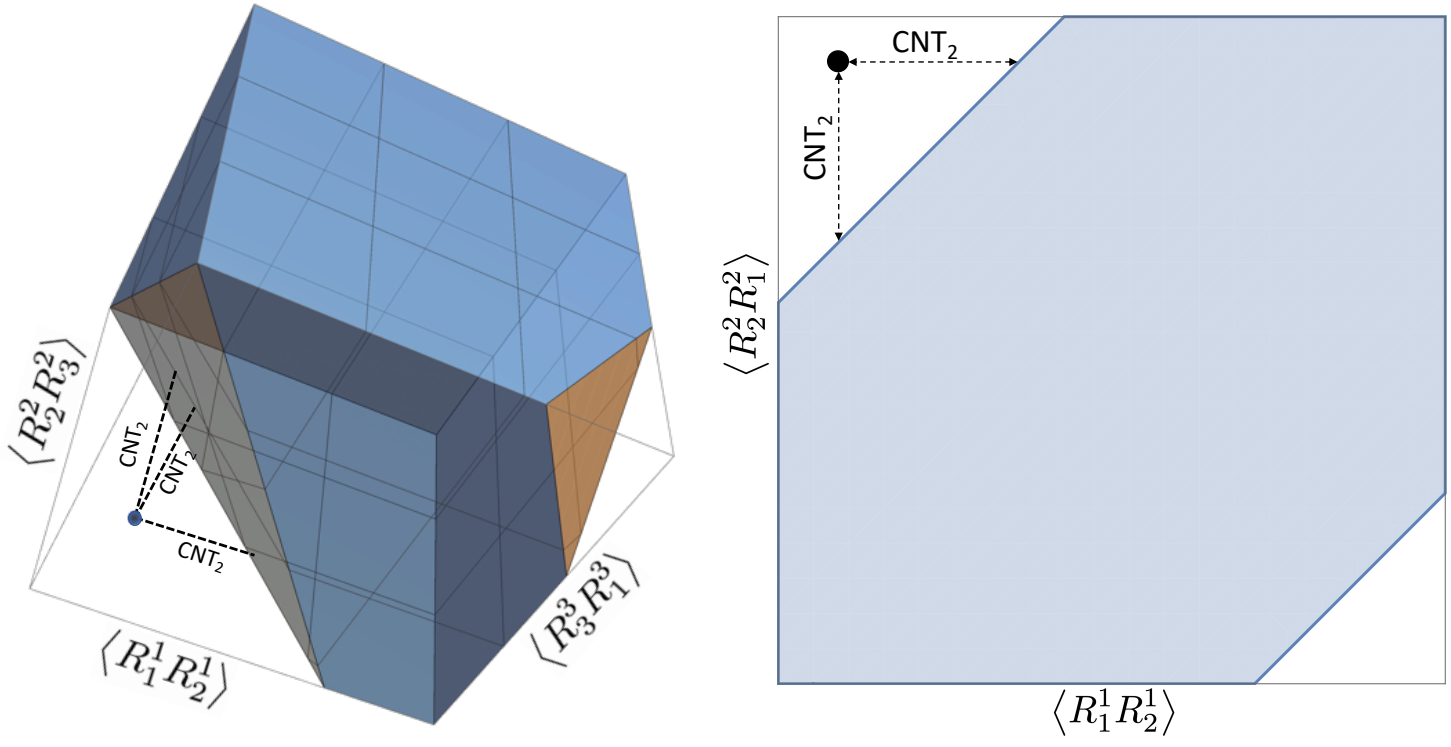


DEGREE OF CONTEXTUALITY, CNT_2





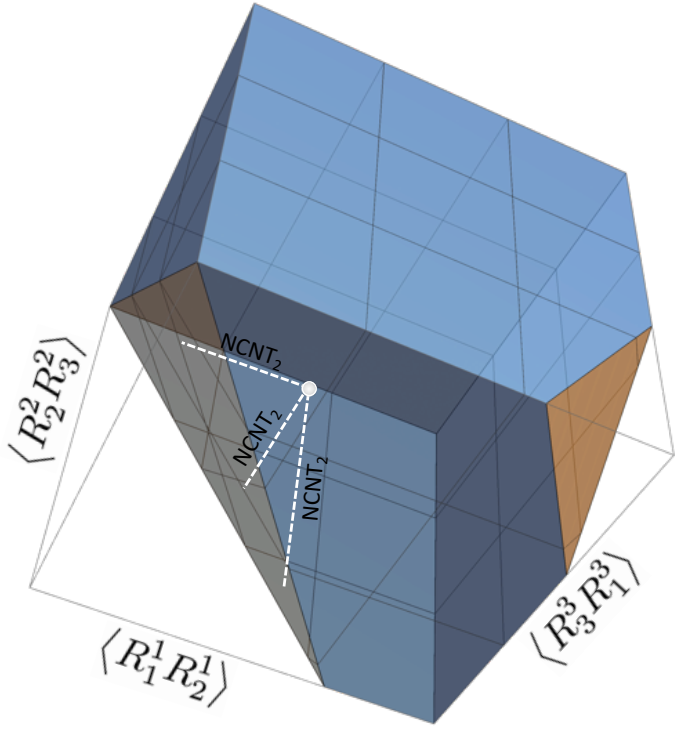
DEGREE OF CONTEXTUALITY, CNT_2

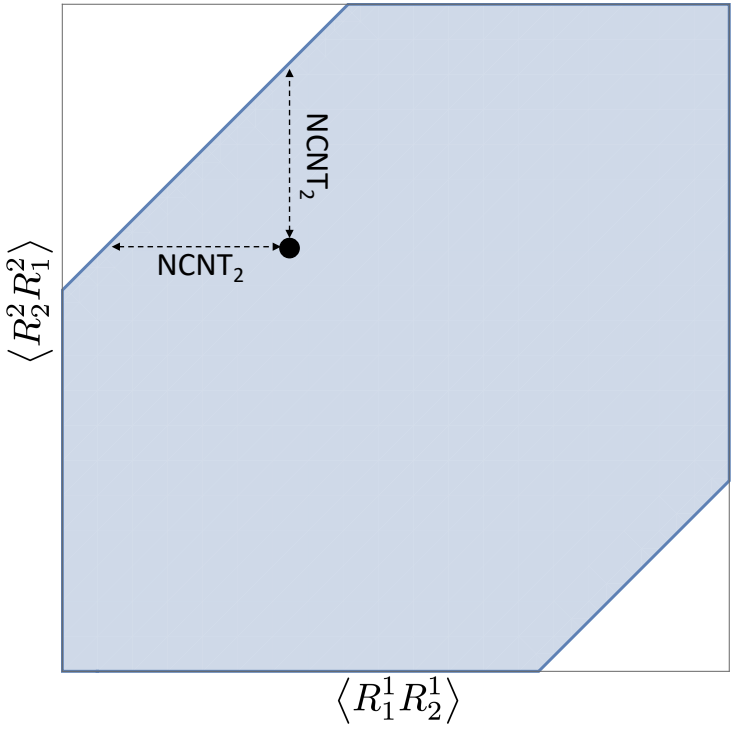


Theorem (Dzh-Kujala-Cervantes, 2020, Phys. Rev. A 101:042119)

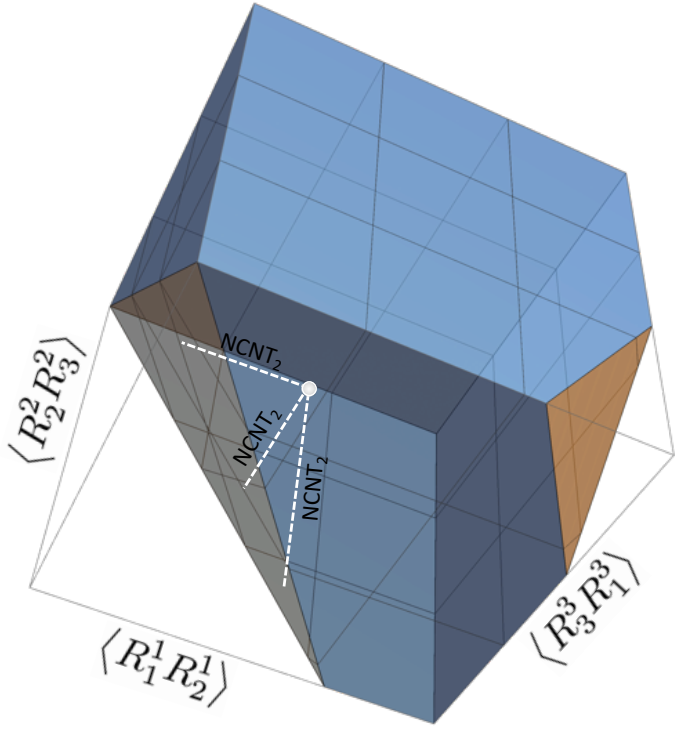
$$\mathrm{CNT}_2 = \frac{1}{4}\left[\max \sum_{\text{odd } \#-} \pm \left\langle \mathrm{R}_i^i \mathrm{R}_{i\oplus 1}^i \right\rangle - \left(n - 2 + \sum_{i=1}^{n} \mathrm{TotVarDis}\left[\mathrm{R}_i^i, \mathrm{R}_i^{i\ominus 1}\right]\right)\right]$$

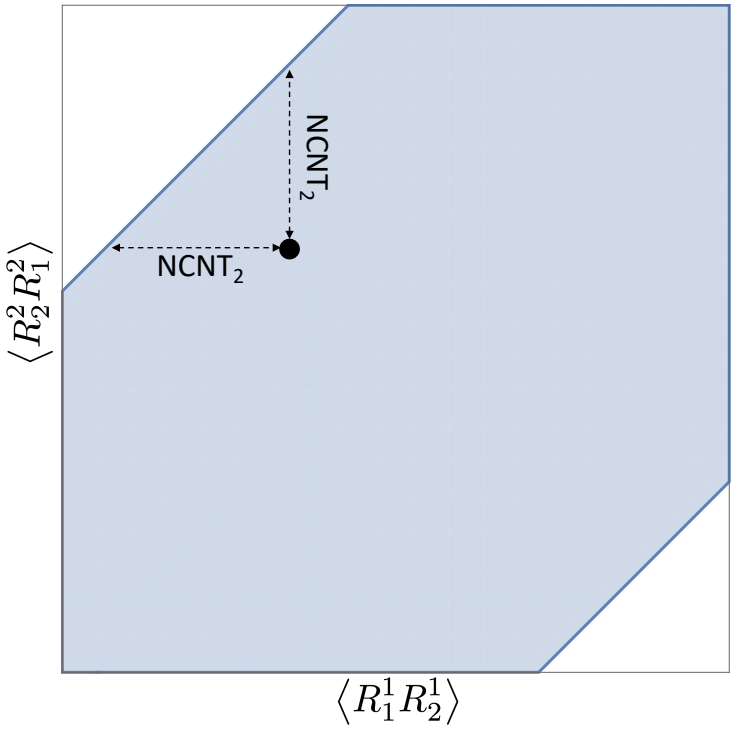
DEGREE OF NONCONTEXTUALITY, NCNT_2





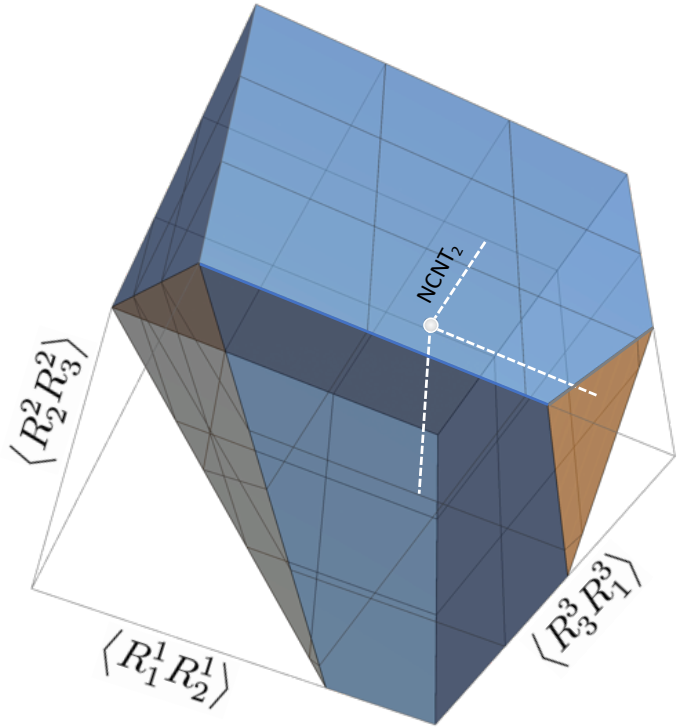
DEGREE OF NONCONTEXTUALITY, NCNT_2

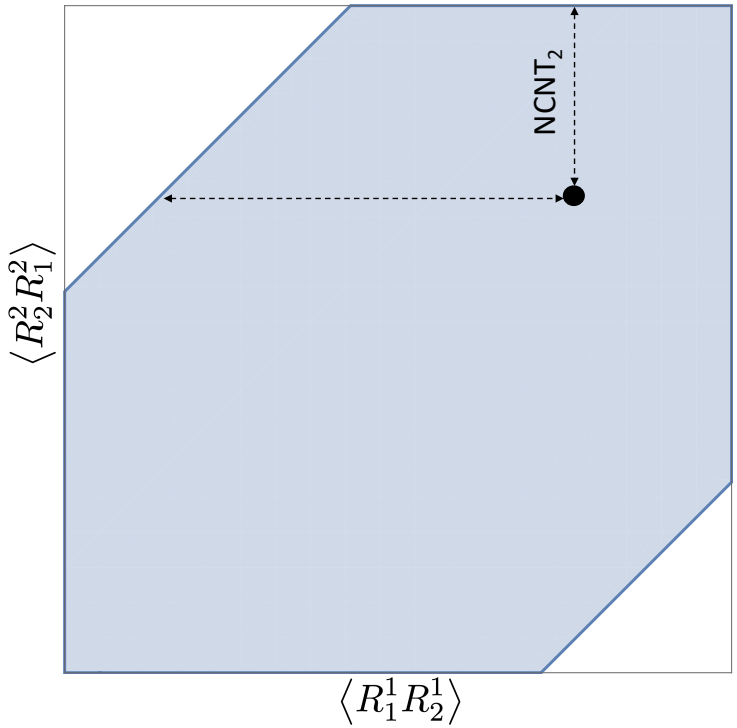




$$\frac{1}{4}\left[\left(n-2+\sum_{i=1}^{n}\mathrm{TotVarDis}\left[R_i^i, R_i^{i\ominus 1}\right]\right)-\max\sum_{\text{odd }\#-}\pm\left\langle R_i^i R_{i\oplus 1}^i\right\rangle\right]$$

DEGREE OF NONCONTEXTUALITY, NCNT_2





Theorem (Dzh-Kujala-Cervantes, 2020, Phys. Rev. A 101:042119)

$$\mathrm{NCNT}_2 = \frac{1}{4}\min\left\{\begin{array}{c}\left(n-2+\sum_{i=1}^{n}\mathrm{TotVarDis}\left[R_i^i, R_i^{i\ominus 1}\right]\right)-\max\sum_{\text{odd }\#-}\pm\left\langle R_i^i R_{i\oplus 1}^i\right\rangle \\ m\left(\left\langle R_i^i R_{i\oplus 1}^i\right\rangle : i=1,\ldots,n\right)\end{array}\right.$$

WHAT IF A SYSTEM IS DETERMINISTIC?

DETERMINISTIC SYSTEM

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

DETERMINISTIC SYSTEM

r_1^1	r_2^1		r_4^1		c^1
r_1^2		r_3^2			c^2
	r_2^3	r_3^3	r_4^3	r_5^3	c^3
		r_3^4		r_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{D}$

DETERMINISTIC SYSTEM: LOGICAL SENTENCES

T	F		F		c^1
F		T			c^2
	F	T	T	F	c^3
		T		F	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{D}$

DETERMINISTIC?

Nico: Zora's claim is true.

Zora: Nico's claim is false.

R_1^1	R_2^1	c^1
R_1^2	R_2^2	c^2
$q_1 =$ "q_2 is true"	$q_2 =$ "q_1 is false"	

DETERMINISTIC?

Nico: Zora's claim is true.

Zora: Max's claim is true.

Max: Nico's claim is false.

R_1^1	R_2^1		c^1
	R_2^2	R_3^2	c^2
R_1^3		R_3^3	c^3
$q_1 =$ "q_2 is true"	$q_2 =$ "q_3 is true"	$q_3 =$ "q_1 is false"	

DETERMINISTIC?

Nico: Zora's claim is true.

Zora: Max's claim is true.

Max: Alex's claim is true.

Alex: Nico's claim is false.

R_1^1	R_2^1			c^1
	R_2^2	R_3^2		c^2
		R_3^3	R_4^3	c^3
R_1^4			R_4^4	c^4
$q_1 =$ "q_2 is T"	$q_2 =$ "q_3 is T"	$q_3 =$ "q_4 is T"	$q_4 =$ "q_1 is F"	

DETERMINISTIC?

R_1^1	R_2^1		c^1
	R_2^2	R_3^2	c^2
R_1^3		R_3^3	c^3
$q_1 =$ "q_2 is true"	$q_2 =$ "q_3 is true"	$q_3 =$ "q_1 is false"	

DETERMINISTIC POSSIBILITIES

1	1	
	1	1
−1		1

−1	−1	
	−1	−1
1		−1

1	1	
	−1	−1
−1		1

1	1	
	1	1
1		−1

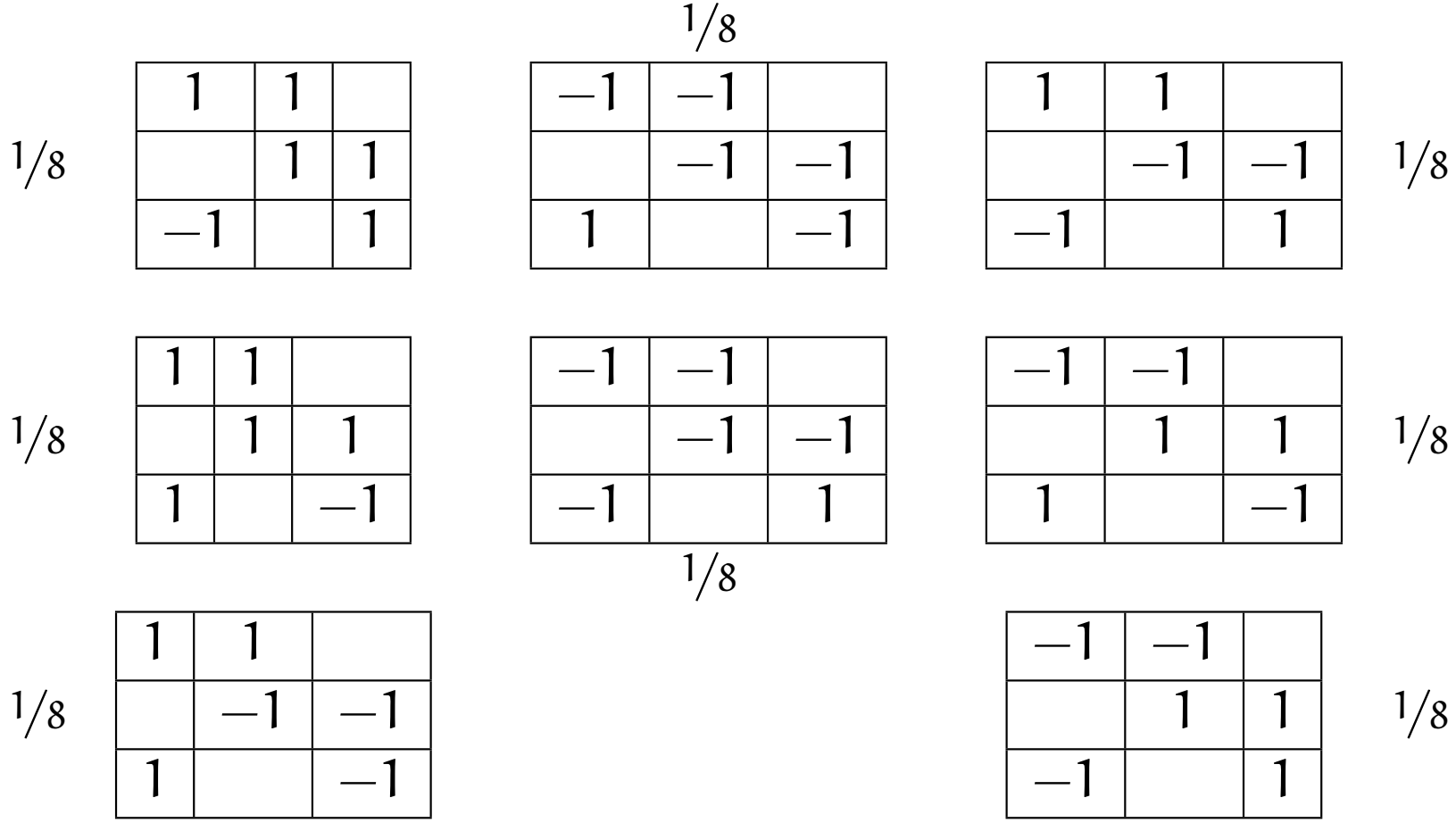
−1	−1	
	−1	−1
1		−1

−1	−1	
	1	1
1		−1

1	1	
	−1	−1
1		−1

−1	−1	
	1	1
−1		1

EPISTEMIC PROBABILITIES



BAYESIAN TREATMENT

$R_1^1 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$	$R_2^1 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$		c^1
	$R_2^2 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$	$R_3^2 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$	c^2
$R_1^3 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$		$R_3^3 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$	c^3
$q_1 =$ "q_2 is true"	$q_2 =$ "q_3 is true"	$q_3 =$ "q_1 is false"	

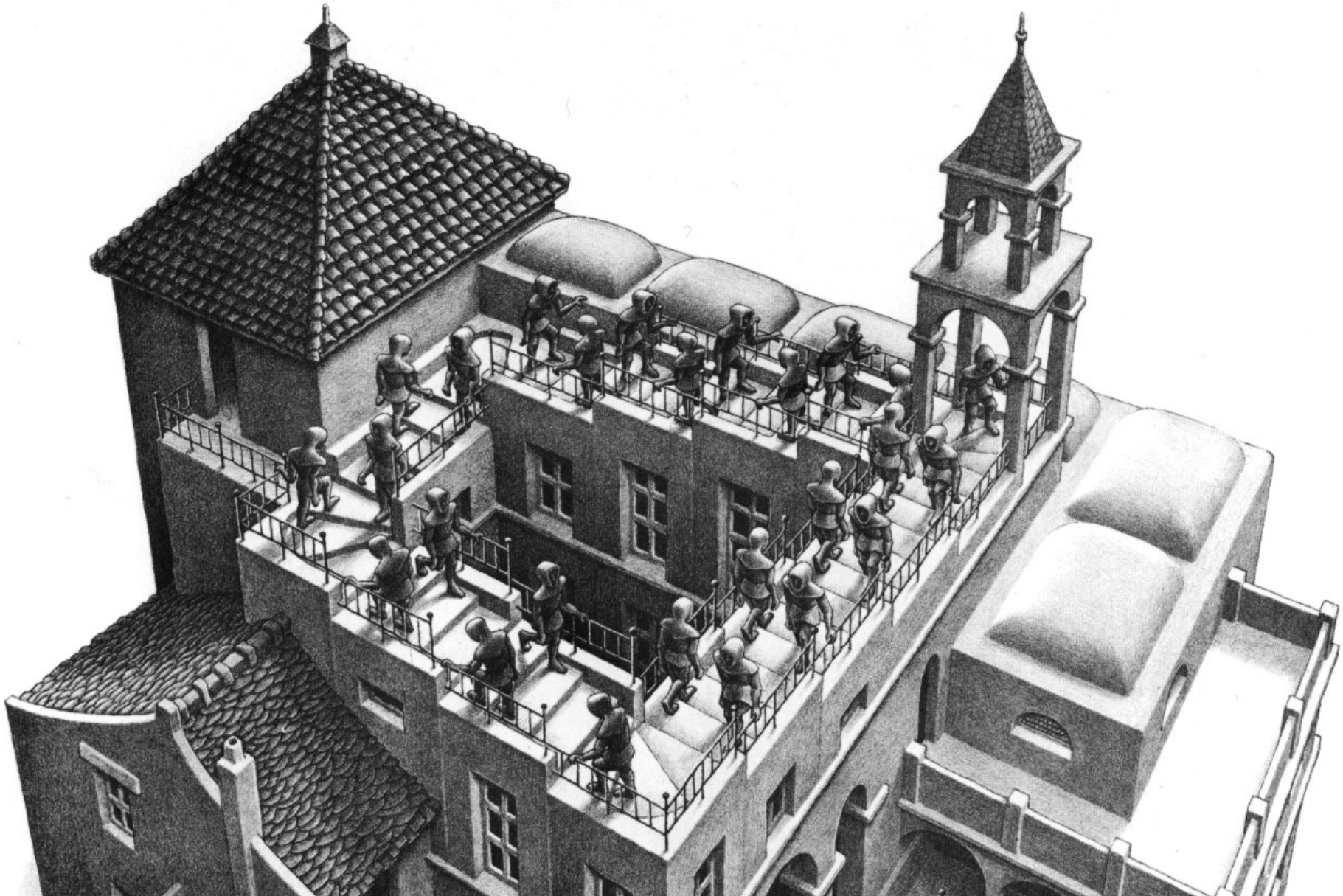
BAYESIAN TREATMENT

$R_1^1 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$	$R_2^1 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$		c^1
	$R_2^2 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$	$R_3^2 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$	c^2
$R_1^3 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$		$R_3^3 = \begin{array}{c\|c} 1 & 1/2 \\ \hline -1 & 1/2 \end{array}$	c^3
$q_1 =$ "q_2 is true"	$q_2 =$ "q_3 is true"	$q_3 =$ "q_1 is false"	

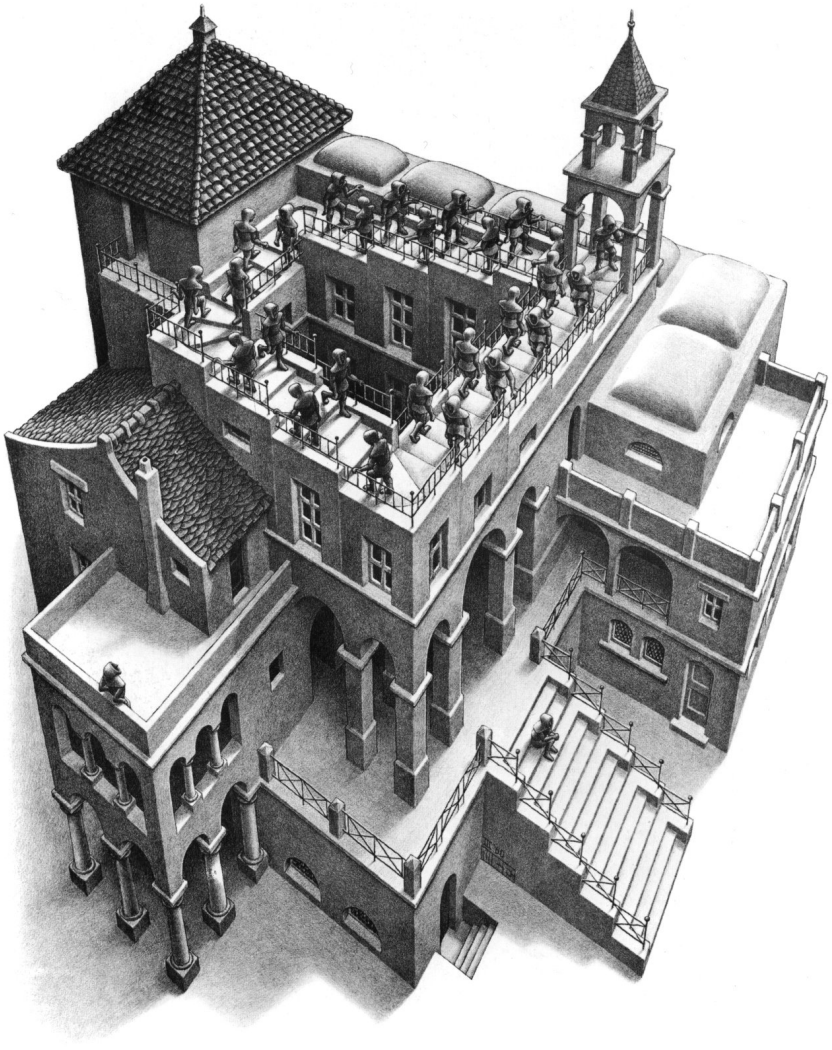
Maximal degree of contextuality: $\mathrm{CNT}_2 = 1/2$

“IMPOSSIBLE” THINGS

M. C. ESCHER “ASCENDING-DESCENDING”

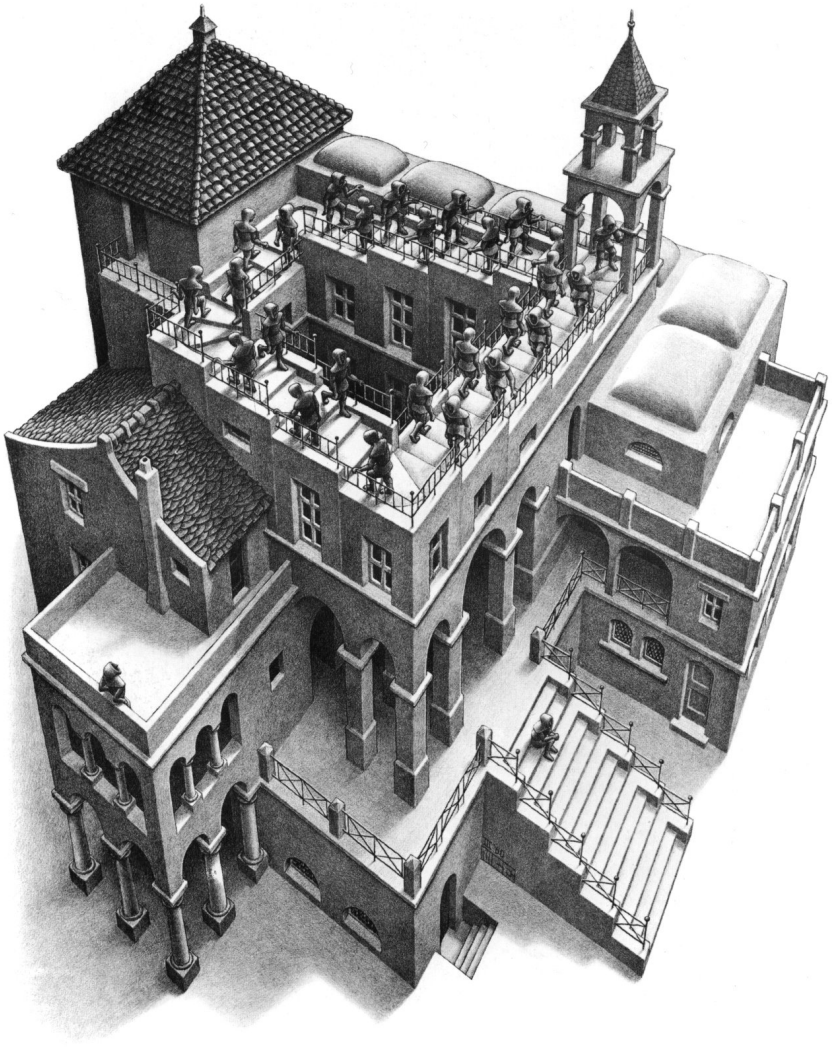


M. C. ESCHER “ASCENDING-DESCENDING”



R_1^1	R_2^1		
	R_2^2	R_3^2	
		R_3^3	R_4^3
R_1^4			R_4^4
q_1	q_2	q_3	q_4

M. C. ESCHER “ASCENDING-DESCENDING”



R_1^1	R_2^1		
	R_2^2	R_3^2	
		R_3^3	R_4^3
R_1^4			R_4^4
R_1^5	R_2^5	R_3^5	R_4^5
q_1	q_2	q_3	q_4

M. C. ESCHER “ASCENDING-DESCENDING”

$R_1^1 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$		$R_2^1 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$					
		$R_2^2 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$		$R_3^2 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$			
				$R_3^3 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$		$R_4^3 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$	
$R_1^3 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$						$R_3^3 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$	

$R_1^5 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$		$R_2^5 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$		$R_3^5 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$		$R_4^5 = \begin{matrix} 1 \\ -1 \end{matrix} \begin{matrix} 1/2 \\ 1/2 \end{matrix}$	

M. C. ESCHER “ASCENDING-DESCENDING”

$R_1^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$	$R_2^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$	$R_3^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$	$R_4^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$

M. C. ESCHER "ASCENDING-DESCENDING"

$R_1^5 = \frac{1}{-1}\,\frac{1/2}{1/2}$	$R_2^5 = \frac{1}{-1}\,\frac{1/2}{1/2}$	$R_3^5 = \frac{1}{-1}\,\frac{1/2}{1/2}$	$R_4^5 = \frac{1}{-1}\,\frac{1/2}{1/2}$

+	+	+	+		−	−	−	−
+	+	+	−		−	−	−	+
+	+	−	+		−	−	+	−
+	−	+	+		−	+	−	−
−	+	+	+		+	−	−	−
+	+	−	−		−	−	+	+
+	−	+	−		−	+	−	+
+	−	−	+		−	+	+	−

M. C. ESCHER “ASCENDING-DESCENDING”

$R_1^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$	$R_2^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$	$R_3^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$	$R_4^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$

$$
\begin{array}{cccc}
+ & + & - & - \\
+ & - & + & - \\
+ & - & - & +
\end{array}
\qquad
\begin{array}{cccc}
- & - & + & + \\
- & + & - & + \\
- & + & + & -
\end{array}
$$

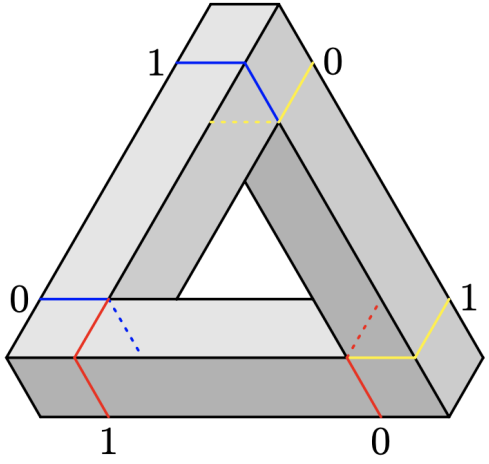
M. C. ESCHER "ASCENDING-DESCENDING"

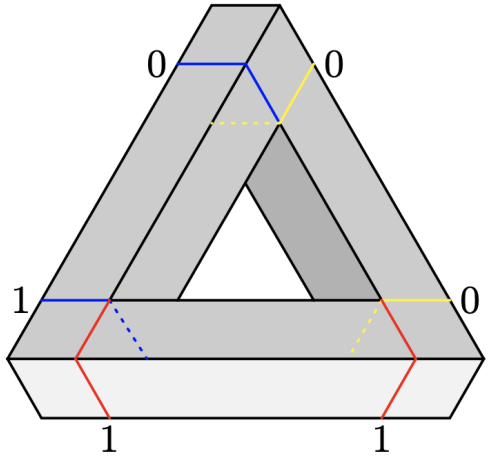
$R_1^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$	$R_2^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$	$R_3^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$	$R_4^5 = \frac{1}{-1}$	$\frac{1/2}{1/2}$

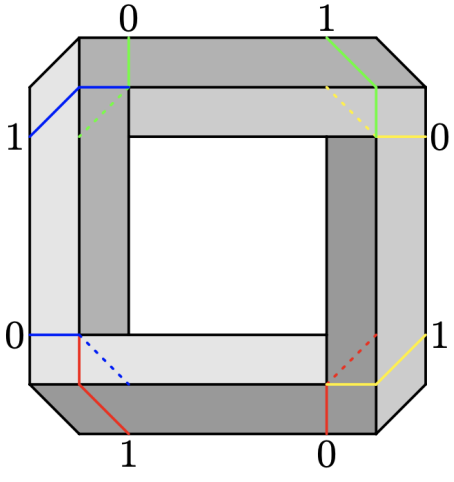
$$
\begin{array}{cccc} + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{array}
\qquad
\begin{array}{cccc} - & - & + & + \\ - & + & - & + \\ - & + & + & - \end{array}
$$

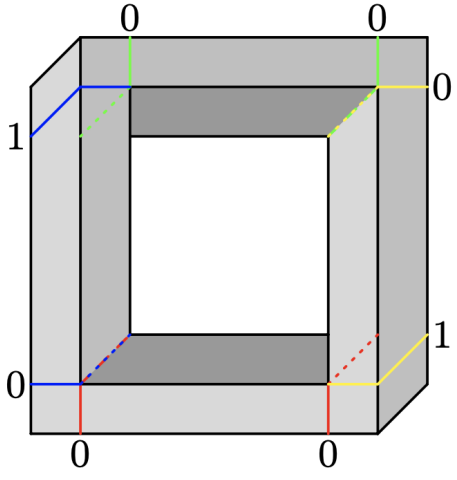
$$\mathrm{CNT}_2 = 4/3$$

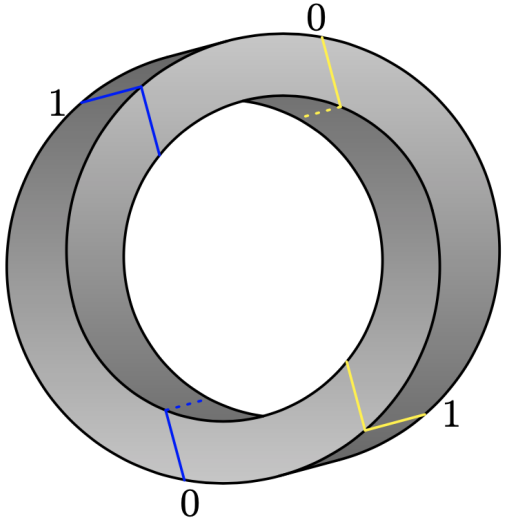
“IMPOSSIBLE FIGURES”



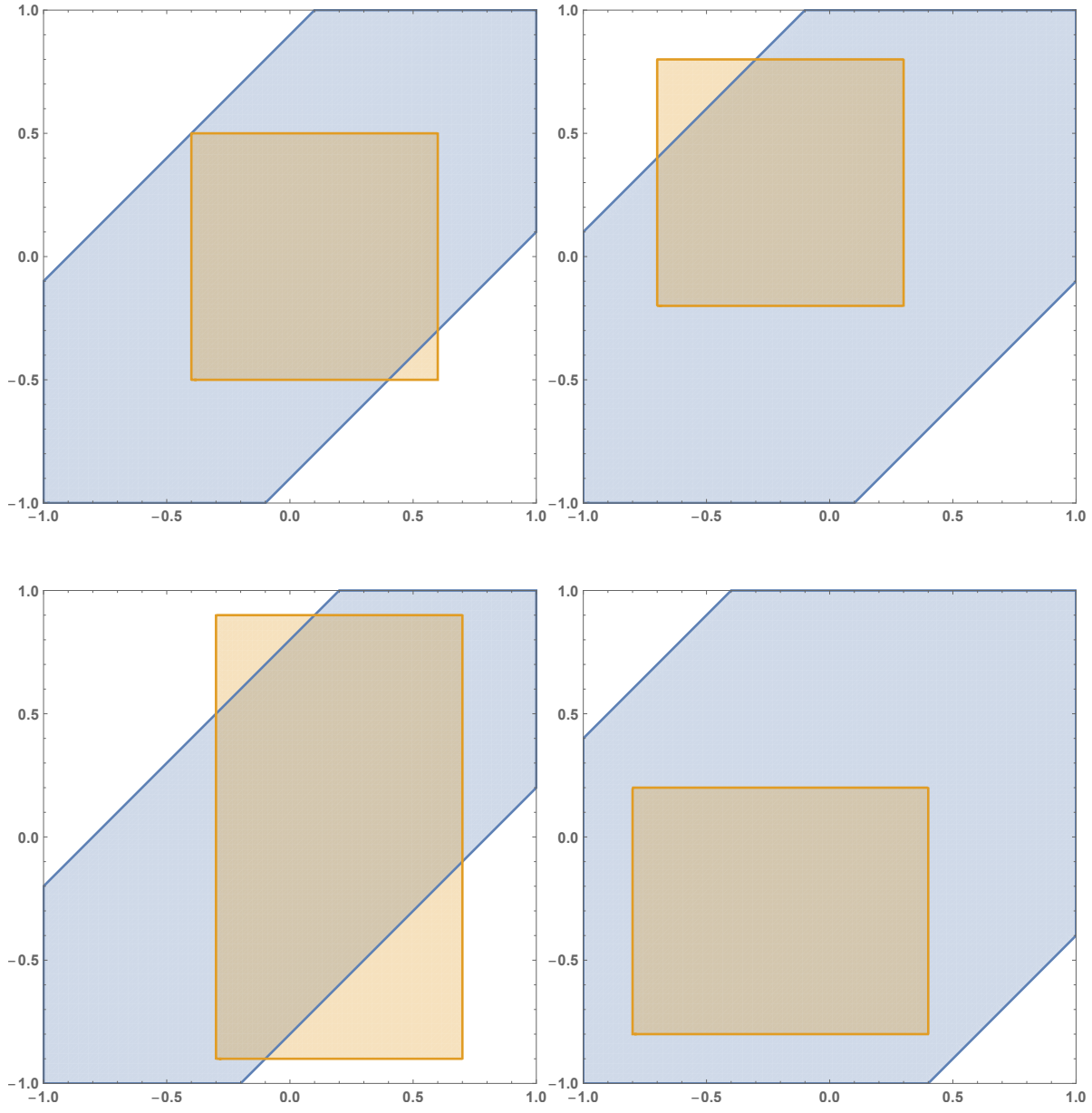




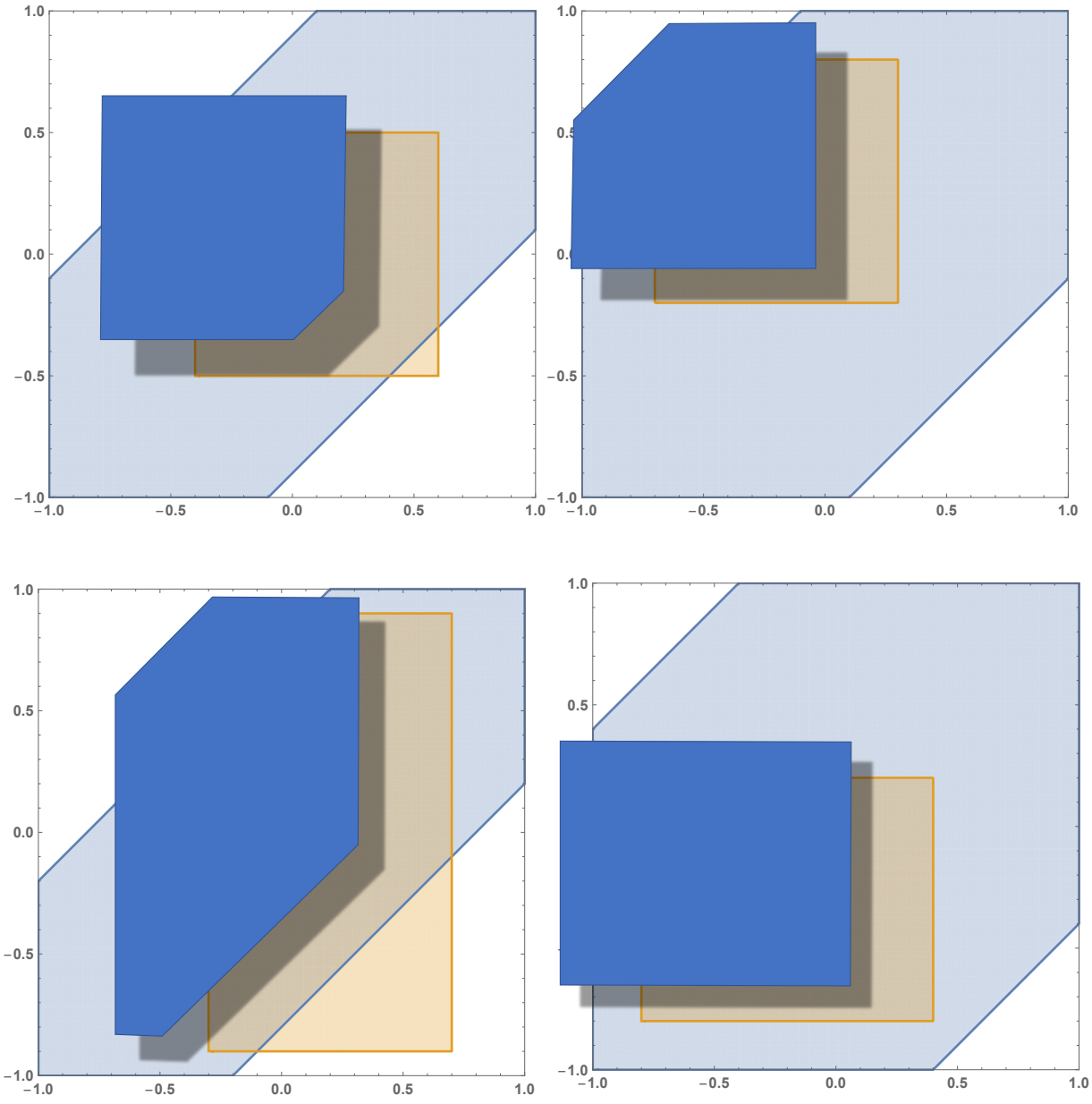




R_1^1	R_2^1
R_1^2	R_2^2

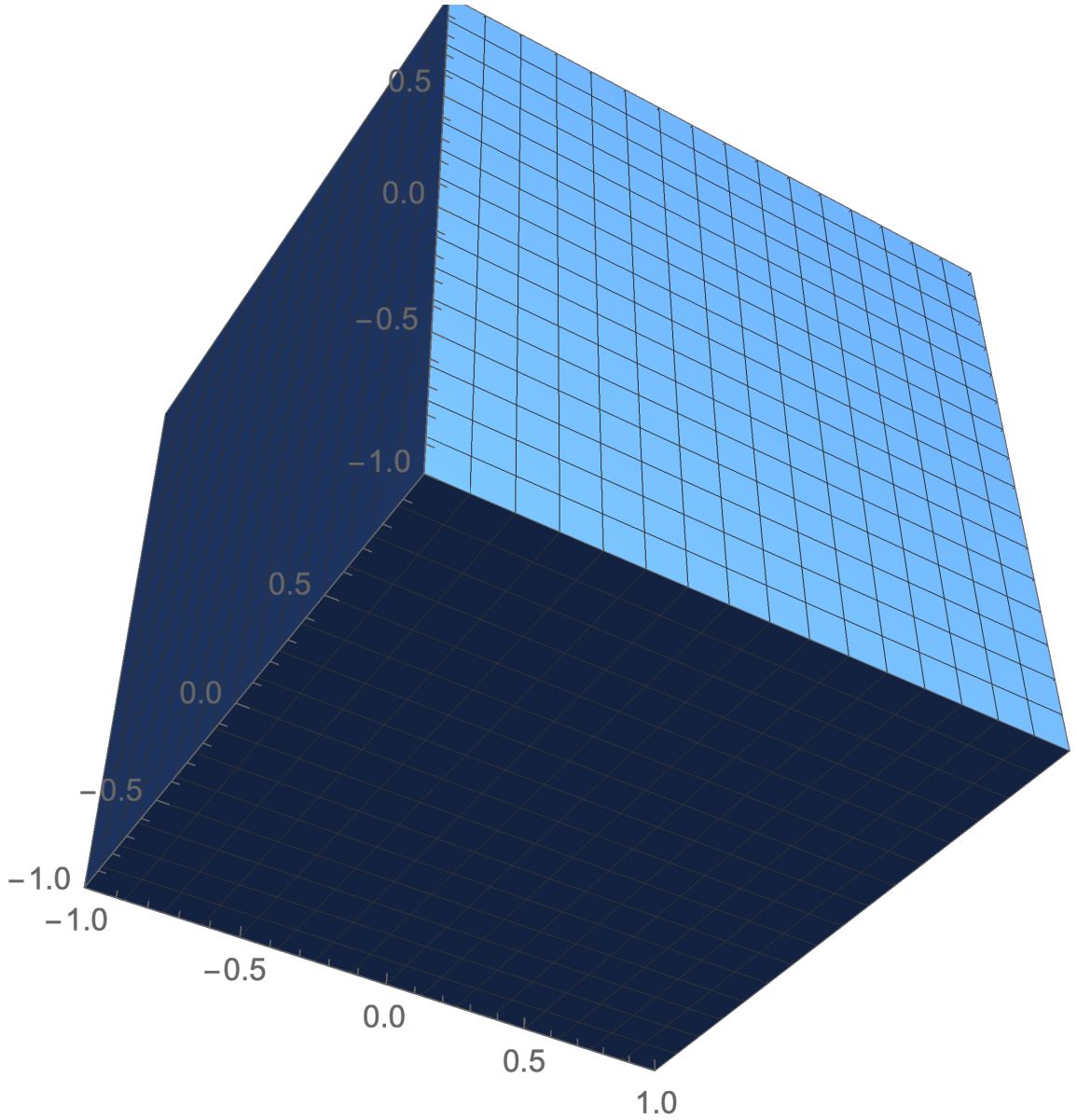


R_1^1	R_2^1
R_1^2	R_2^2

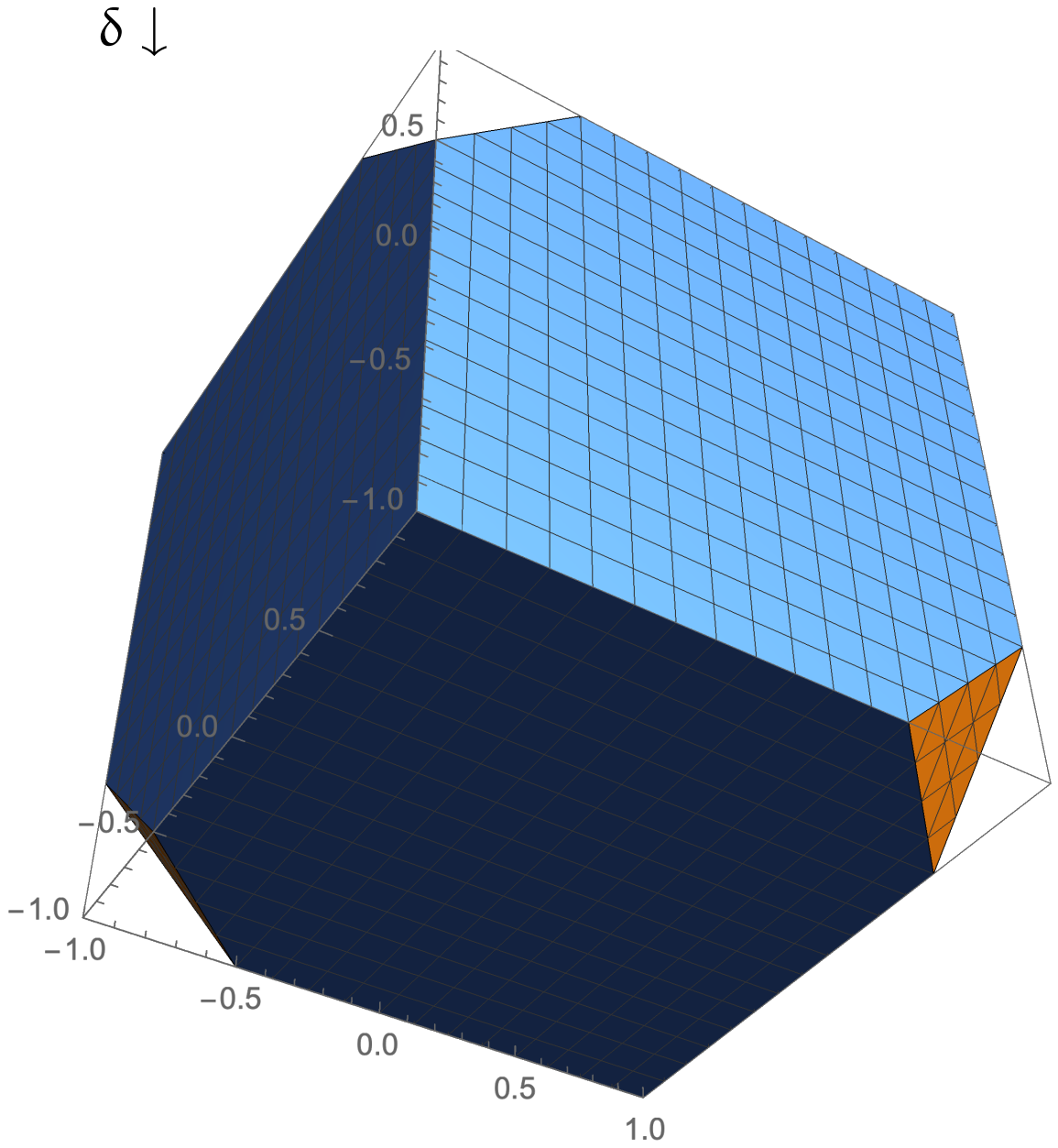


R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3

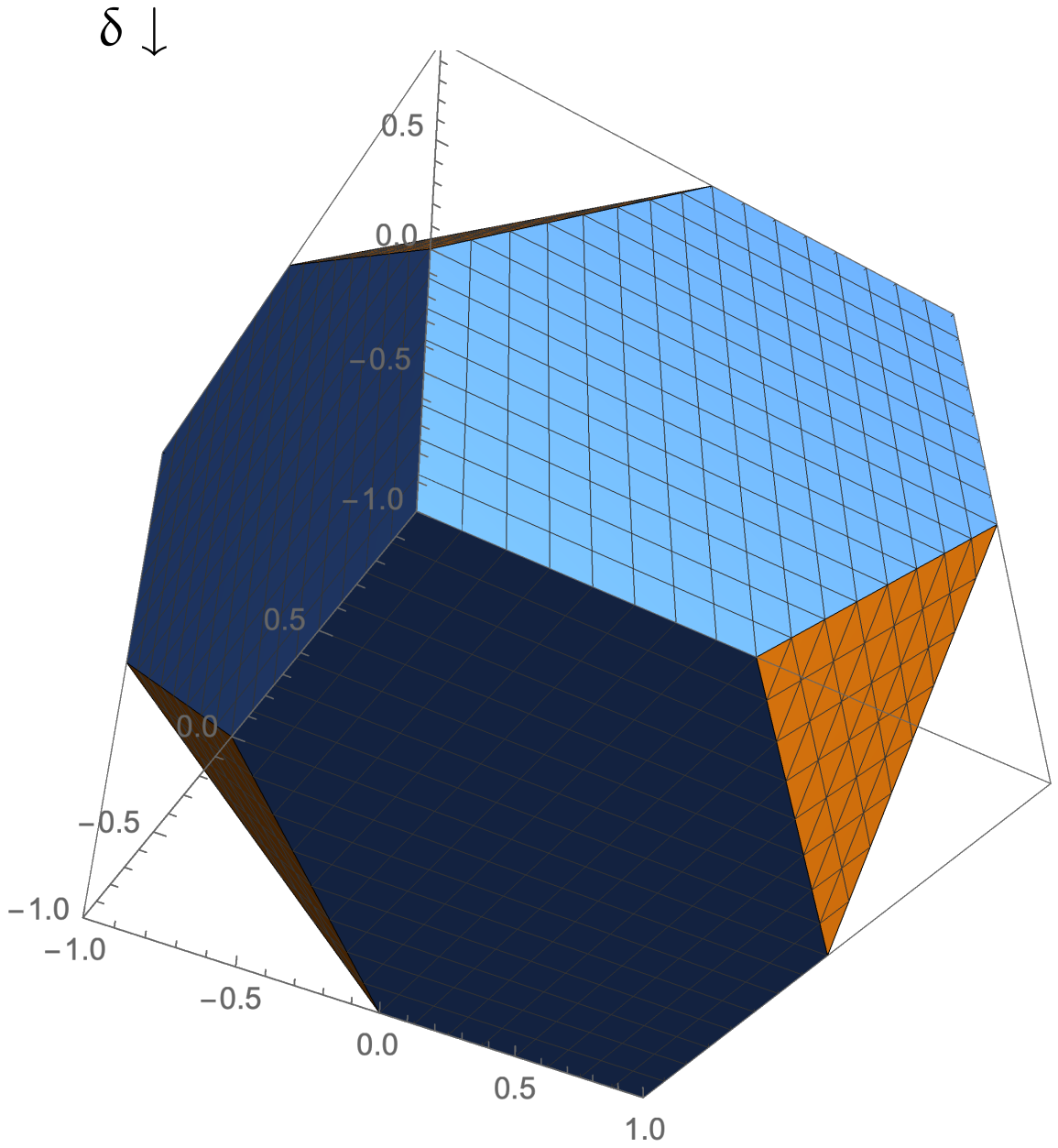
$$\delta = \max$$



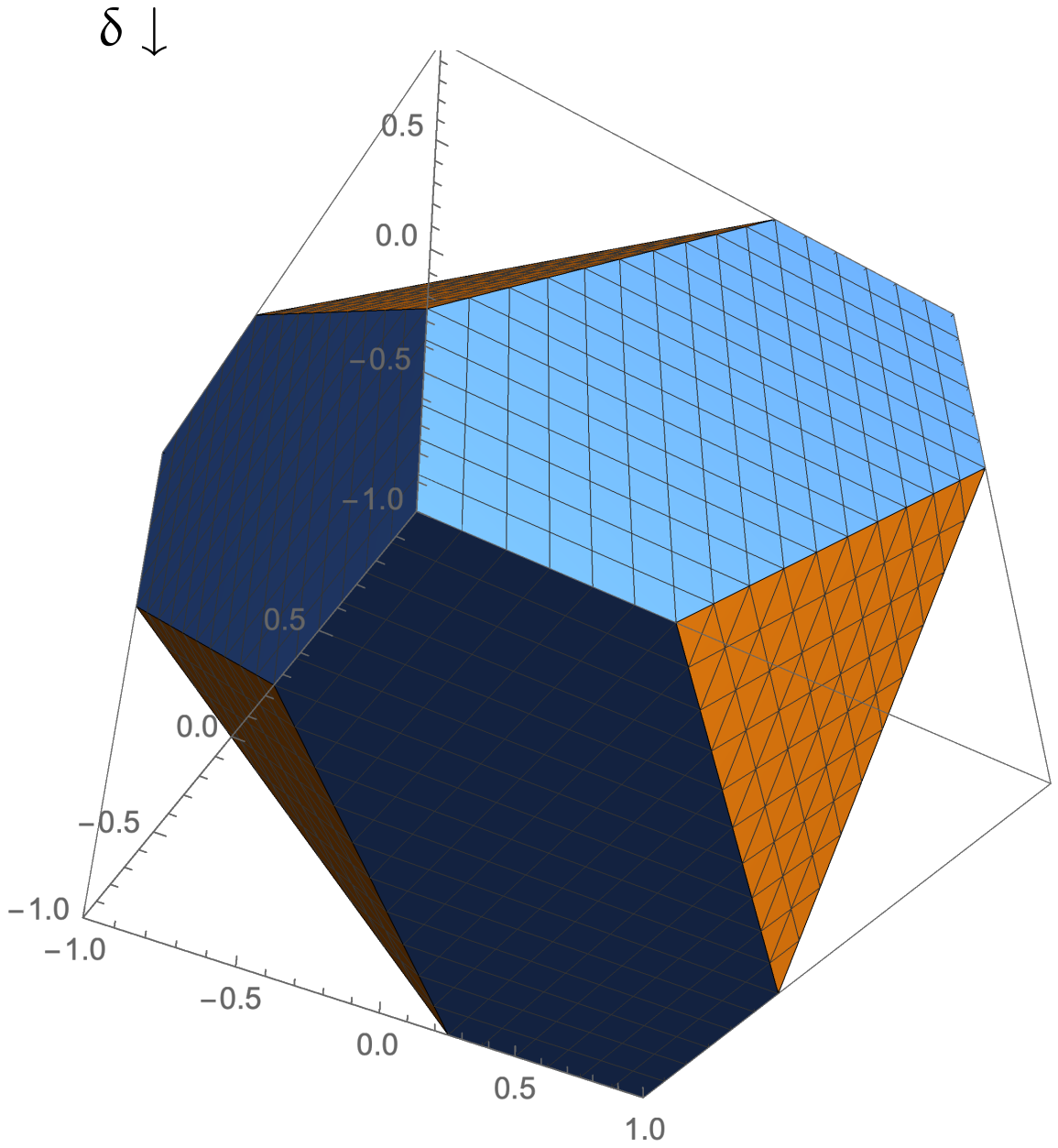
R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3



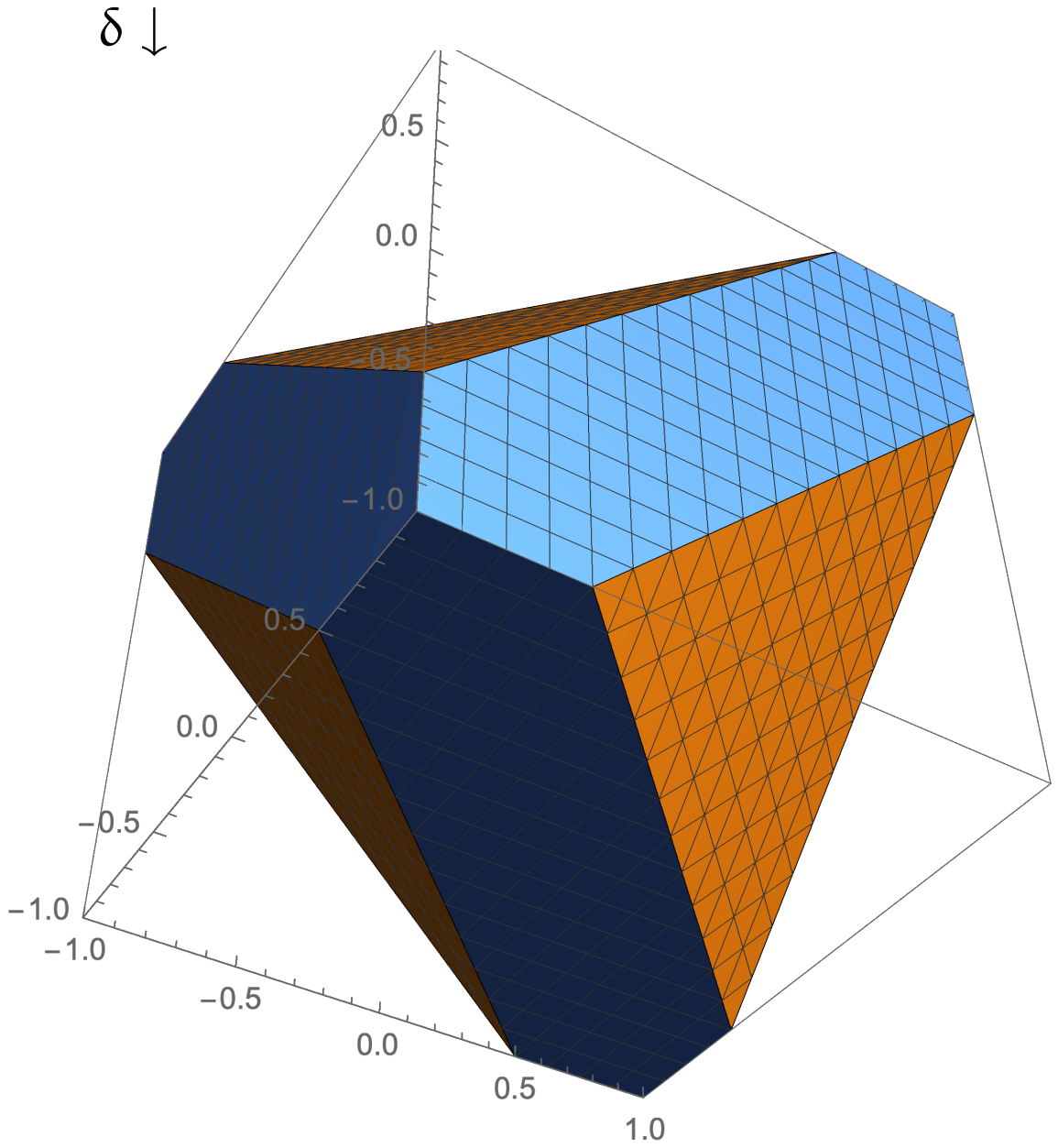
R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3



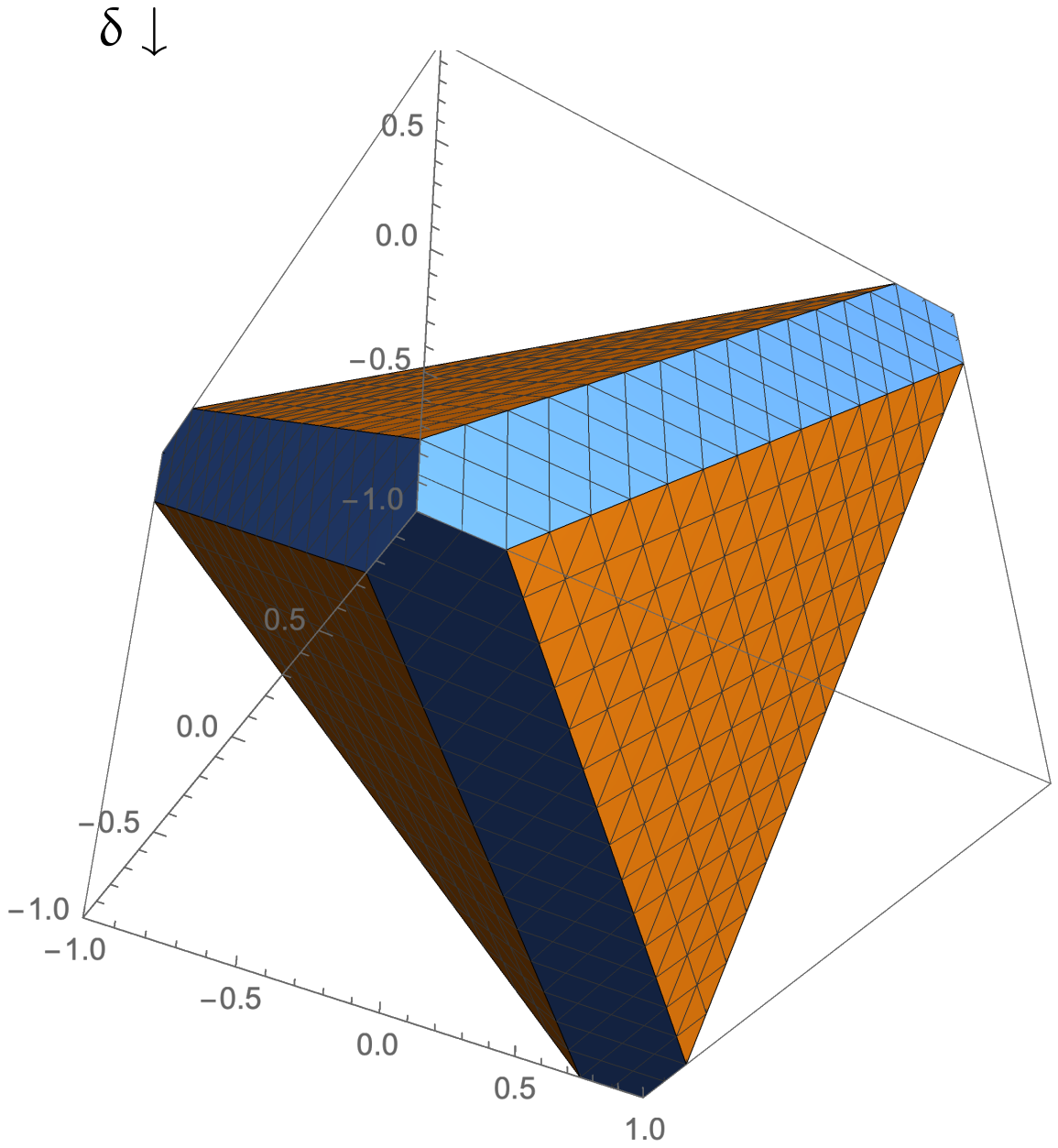
R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3



R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3

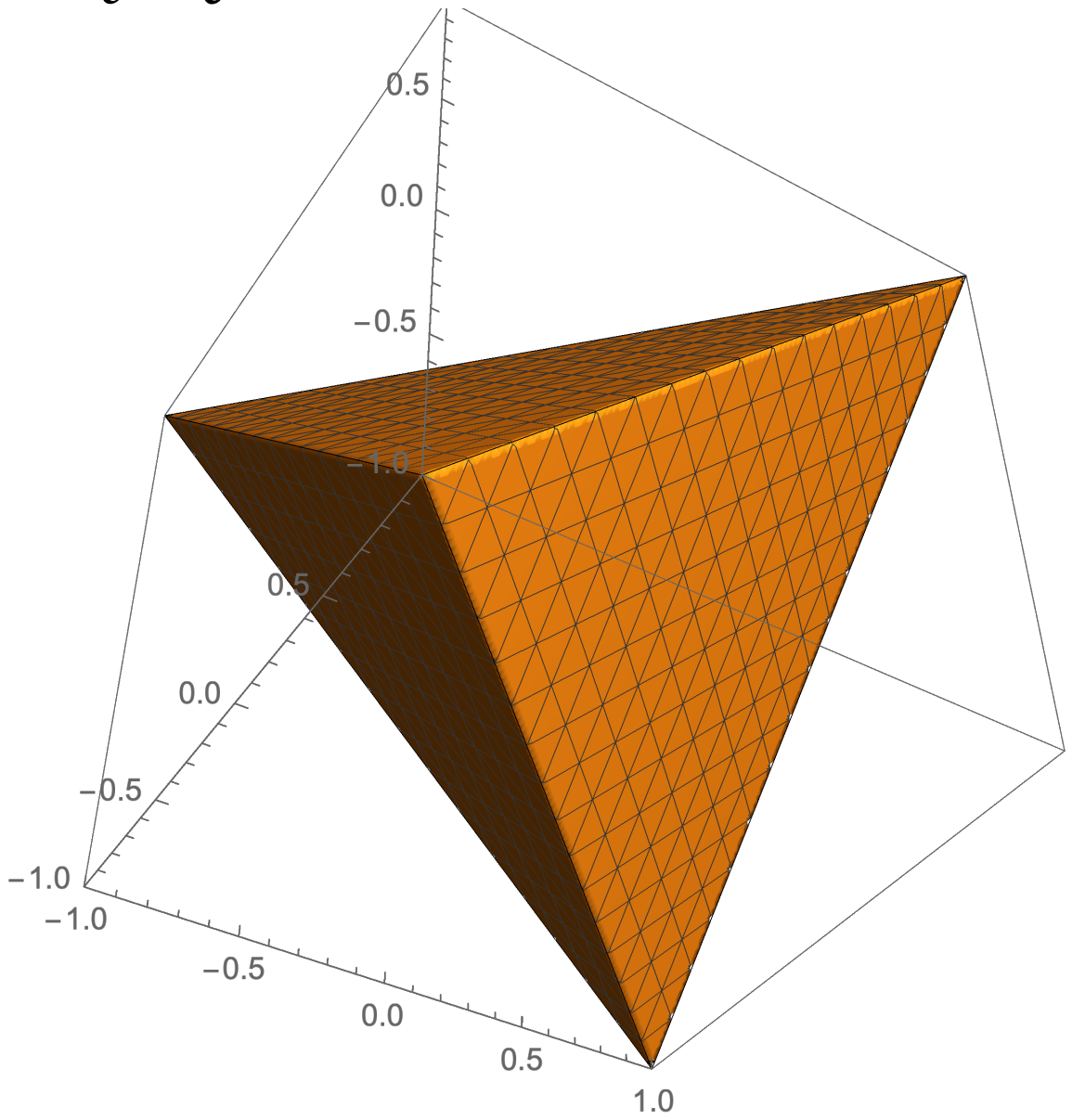


R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3



R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3

$\delta = 0$



R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3